1. Consider the function
\[ \varphi(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-\epsilon^2}\right), & |x| < \epsilon \\ 0, & |x| \geq \epsilon. \end{cases} \]
You will find \( \varphi \in D(\mathbb{R}^n) \) and has the desired properties.

2. The remainder is
\[ R_N(x, h) = \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x + th) \, dt, \]
For \( h \in \mathbb{R}^n \) fixed, it is clear that \( R_N(\cdot, h) \) vanishes unless \( x + th \in \text{supp}(\varphi) \) for some \( t \in [0, 1] \). Hence
\[ \text{supp}(R_N) \subset \text{supp}(\varphi) + B_{|h|}. \]
Choose \( |h| \) small enough so that the left hand side is contained in some compact subset of \( X \). Also
\[ \sup_x |\partial^\alpha_x R_N(x, h)| \lesssim \sum_{|\beta|=N+1} \frac{|h_1|^{\beta_1} \cdots |h_n|^{\beta_n}}{\beta!} \sup_x \int_0^1 |\partial^{\alpha+\beta} \varphi(x + th)| \, dt \]
\[ \lesssim \sum_{|\beta|=N+1} \frac{|h_1|^{\beta_1} \cdots |h_n|^{\beta_n}}{\beta!} \]
\[ \lesssim \left( |h_1| + \cdots + |h_n| \right)^{N+1} \]
\[ \lesssim \left[ n^{1/2} \left( h_1^2 + \cdots + h_n^2 \right)^{1/2} \right]^{N+1} \]
\[ \lesssim |h|^{N+1} \]
where in the second to last line we used Cauchy-Schwarz. The implied constant in this final estimate will depend on \( \{\alpha, N, n, \varphi\} \). We see \( \partial^\alpha R_N(x, h) = o(|h|^N) \) uniformly in \( x \).

3. None other than \( \varphi = 0 \). For the keen: We shall prove that there are no non-zero analytic test functions. Initially assume \( X \) is connected and assume \( \varphi \in D(X) \) is analytic (i.e. locally it can be identified with its Taylor series). Since \( \varphi \in D(X) \) there is an open set \( U \subset X \) on which \( \varphi = 0 \). Let \( E \subset X \) be the interior of the set on which \( \varphi = 0 \). Then \( E \) is open and non-empty since \( U \subset E \). We will show \( E \) is relatively closed in \( X \). Take any point \( x \in X \cap \overline{E} \). By our hypothesis, we know there is a \( \delta > 0 \) such \( \varphi \) converges to its Taylor series on the ball \( B_{\delta}(x) \)
\[ \varphi(y) = \sum_{|\alpha| \geq 0} \frac{(y-x)^\alpha}{\alpha!} \partial^\alpha \varphi(x), \quad y \in B_{\delta}(x). \]
Now take \( x' \in E \) so that \( x \in B_{\delta'}(x') \) for some \( \delta' > 0 \). Since \( \varphi = 0 \) on \( E \), we must have
\[ 0 = \sum_{|\alpha| \geq 0} \frac{(y-x)^\alpha}{\alpha!} \partial^\alpha \varphi(x), \quad y \in B_{\delta}(x) \cap B_{\delta'}(x'). \]
Consequently \( \varphi(x) = 0 \). Hence \( x \in E \) and we deduce that \( E \) is relatively closed in \( X \). Since \( E \) is also open in \( X \), we have \( E = X \) by connectedness. So \( \varphi = 0 \) identically in \( X \). The connectedness of \( X \) is artificial, since we can just extend our test functions by zero and interpret them as elements of \( D(\mathbb{R}^n) \).
4. Choose $\epsilon > 0$ sufficiently small so that $|x - y| \leq 4\epsilon$ when $x \in K$ and $y \in X^c = \mathbb{R}^n \setminus X$. Following the hint, we introduce an indicator function on a neighbourhood of $K$: let $\chi = 1$ on the set

$$K_{2\epsilon} = \{ y : |x - y| \leq 2\epsilon \text{ for some } x \in K \}$$

and $\chi = 0$ otherwise. Now fix $\varphi \in D(X)$ with $\int \varphi \, dx = 1$ and $\text{supp}(\varphi) \subset B_1$. It follows that $\varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon)$ is an approximation to the identity as $\epsilon \downarrow 0$ and $\text{supp}(\varphi_\epsilon) \subset B_\epsilon$. We consider the function

$$\phi(x) = (\chi * \varphi_\epsilon)(x) = \int \chi(y) \varphi_\epsilon(x - y) \, dy.$$ 

Certainly $\phi \in C^\infty(X)$ and it is clear that

$$\text{supp}(\phi) \subset \text{supp}(\chi) + \text{supp}(\varphi_\epsilon) \subset K_{2\epsilon} + B_\epsilon \subset K_{3\epsilon}$$

so $\phi \in D(X)$. Since $\int \varphi_\epsilon \, dx = 1$ we have

$$1 - \phi(x) = \int \varphi_\epsilon(y) \left(1 - \chi(x - y)\right) \, dy$$

which vanishes if $x - y \in K_{2\epsilon}$ for all $y \in B_\epsilon$. So $\phi = 1$ in $K_\epsilon$ and we are done.

5. By definition: for each compact $K \subset X$ there exist constants such that

$$|\langle \partial^\alpha T, \varphi \rangle| \overset{\text{def}}{=} |(-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle| \leq C \sum_{|\beta| \leq N} \sup |\partial^{\alpha + \beta} \varphi| \leq C \sum_{|\gamma| \leq N + |\alpha|} \sup |\partial^\gamma \varphi|$$

for each $\varphi \in D(X)$ with $\text{supp}(\varphi) \subset K$. So $\partial^\alpha T \in \mathcal{D}'(X)$ and if $\text{ord}(T) = m$ then $\text{ord}(\partial^\alpha T) \leq m + |\alpha|$. To construct a distribution for which this inequality is strict for a particular multi-index, consider the linear map $\langle T, \varphi \rangle = \int \partial^\alpha \varphi(x_1, 0, \ldots, 0) \, dx_1$ and calculate $\partial T / \partial x_i$ for $i = 1$ and $i \neq 1$.

6. Using the definitions we have

$$\langle (fT'), \varphi \rangle = -\langle fT, \varphi' \rangle = -\langle T, f\varphi' \rangle = -\langle T, (f\varphi)' \rangle + \langle T, f' \varphi \rangle = \langle (fT' + f'T), \varphi \rangle$$

i.e. $fT' = fT + f'T$ in $\mathcal{D}'(X)$. The general case is similar.

7. If $\varphi \in D(X)$ then there are only a finite number of the $x_k$ contained in $\text{supp} \varphi$ (if there were infinitely many inside the compact set $\text{supp} \varphi$, we would be able to go to a subsequence which converges, contradicting the fact that the sequence has no limit points). Hence

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^\infty \partial^\alpha \varphi(x_k) = \sum_{k \in I} \partial^\alpha \varphi(x_k)$$

where $I = \{ k : x_k \in \text{supp} \varphi \}$ is a finite set that depends only on $\text{supp} \varphi$. So the linear map is well-defined. To see that it’s a distribution: fix a compact $K \subset X$, and take $\varphi \in D(X)$ with $\text{supp} \varphi \subset K$. Then

$$|\langle u_\alpha, \varphi \rangle| \leq \sum_{k \in I} |\partial^\alpha \varphi(x_k)| \leq \sum_{k \in I} \sup |\partial^\alpha \varphi| \leq |I| \sum_{|\beta| \leq |\alpha|} \sup |\partial^\beta \varphi|.$$ 

Since $|I|$ is dependent only on $K$, we deduce that $u_\alpha \in \mathcal{D}'(X)$ for all multi-indices $\alpha$ and clearly $\text{ord}(u_\alpha) \leq |\alpha|$. To show that the order is equal to $|\alpha|$, set $\varphi_m(x) = (x - x_1)^\alpha \psi(m(x - x_1))$ for some $\psi \in D(\mathbb{R}^n)$ with sufficiently small support and $\psi(0) = 1$. Then $\sup |\partial^\beta \varphi_m| \lesssim m_{|\beta|-|\alpha|}$, since

$$\partial^\beta \varphi_m(x) = \sum_{|\gamma| \leq |\beta|} \binom{\beta}{\gamma} \partial^{\beta - \gamma} (x - x_1)^\alpha m^{\alpha |\gamma|} \partial^{\gamma} \psi(m(x - x_1)),$$

and the right hand side vanishes unless $|x - x_1| \lesssim 1/m$. So $\sup |\partial^\beta \varphi_m| \to 0$ as $m \to \infty$ if $|\beta| < |\alpha|$. But $\langle u_\alpha, \varphi_m \rangle = |\alpha|! \geq 1$ so the $\text{ord}(u_\alpha)$ cannot be less than $|\alpha|$. For intuition: our sequence of functions looks exactly like $(x - x_1)^\alpha$ localised closer and closer around the point $x = x_1$. 


8. (a) If \( u' = 1 \) then \( (u - x)' = 0 \). In lectures we showed that \( v' = 0 \) in \( D'(\mathbb{R}) \). Hence the general solution to \( u' = 1 \) is \( u = x + \text{const} \).

(b) Try \( u = c \delta_0 \) for a particular solution. Then

\[
\langle xu', \varphi \rangle = -\langle u, (x \varphi)' \rangle = -\langle u, \varphi \rangle - \langle u, x \varphi' \rangle = -c \langle \delta_0, \varphi \rangle.
\]

So we find \( xu' = \delta_0 \) by setting \( u = -\delta_0 \). Now we need to classify all solutions to the homogeneous problem. Consider equation \( xu = 0 \) in \( D'(\mathbb{R}) \). Introducing \( \rho \in D(\mathbb{R}) \) with \( \rho = 1 \) in a neighbourhood of \( \{0\} \) find

\[
\langle v, \varphi \rangle = \langle v, \rho \varphi \rangle + \langle v, (1 - \rho) \varphi \rangle.
\]

By Taylor’s theorem

\[
\varphi(x) = \varphi(0) + x \tilde{\varphi}(x), \quad (1 - \rho(x)) \varphi(x) = x \tilde{\varphi}(x)
\]

where \( \tilde{\varphi} \in C^\infty(\mathbb{R}) \) and \( \tilde{\varphi} \in D(\mathbb{R}) \). And so

\[
\langle v, \varphi \rangle = \langle v, \rho \varphi \rangle + \langle v, 1 \rangle \langle \rho, \varphi \rangle = c_0 \langle \delta_0, \varphi \rangle
\]

where \( c_0 = \langle v, \rho \rangle \). We conclude that the most general solution to \( xu = 0 \) in \( D'(\mathbb{R}) \) is \( u = c_0 \delta_0 \). It follows that the most general solution to \( xu' = 0 \) is \( u = c_0 H + c_1 \), where \( H \) is the Heaviside function which satisfies \( H' = \delta_0 \). So the general solution to \( xu' = \delta_0 \) in \( D'(\mathbb{R}) \) is \( u = c_0 H + c_1 - \delta_0 \).

(c) It is clear that \( \text{supp}(u') \subset \mathbb{Z} \). Indeed, if \( \text{supp}(\varphi) \cap \mathbb{Z} = \emptyset \) then

\[
\langle u', \varphi \rangle = \langle (e^{2\pi i x} - 1)u', (e^{2\pi i x} - 1)^{-1} \varphi \rangle = 0
\]

since \( (e^{2\pi i x} - 1)^{-1} \) is smooth on \( \text{supp}(\varphi) \). Introduce the functions \( \rho_n \in D(\mathbb{R}) \)

\[
\rho_n(x) = \begin{cases} 
1, & |x - n| \leq \epsilon \\
0, & |x - n| > 2\epsilon 
\end{cases}
\]

where \( \epsilon < 1/10 \) say (so \( \text{supp}(\rho_n) \cap \text{supp}(\rho_m) = \emptyset \) for \( n \neq m \)). Then we have

\[
0 = \langle (e^{2\pi i x} - 1)u', \varphi \rangle = \sum_n \langle (e^{2\pi i x} - 1)u'_n, \varphi \rangle = \sum_n \langle v_n, \varphi \rangle
\]

where \( u'_n = \rho_n u' \) and \( v_n \) is defined accordingly. Now since the supports of the \( v_n \) are disjoint and \( \sum v_n = 0 \) it must be the case that \( v_n = 0 \) for each \( n \). Indeed, we have

\[
\langle v_n, \varphi \rangle = \langle v_n, \rho_n \varphi \rangle = -\sum_{m \neq n} \langle v_m, \rho_n \varphi \rangle = 0
\]

since \( \text{supp}(\rho_n \varphi) \cap \text{supp}(v_m) = \emptyset \) for \( m \neq n \). So our problem is reduced to solving

\[
(e^{2\pi i x} - 1)u'_n = 0.
\]

We may concentrate on \( m = 0 \) since the other cases are just translations of this. We have

\[
\langle u'_0, \varphi \rangle = \langle u'_0, \rho_0 \varphi \rangle
\]

and by Taylor’s theorem \( \varphi(x) = \varphi(0) + x \tilde{\varphi}(x) \) for some \( \tilde{\varphi} \in C^\infty(\mathbb{R}) \). Note that we can write

\[
(e^{2\pi i x} - 1) = xf(x)
\]

where \( f \in C^\infty(\mathbb{R}) \) and \( f > 0 \) on \( \text{supp}(\rho_0) \). Hence

\[
\langle u'_0, \varphi \rangle = \langle (e^{2\pi i x} - 1)u'_0, \varphi \rangle = c_0 \langle \delta_0, \varphi \rangle + \left( (e^{2\pi i x} - 1)u'_0, \frac{\rho_0 \varphi}{f} \right) = c_0 \langle \delta_0, \varphi \rangle.
\]

We deduce \( u_0 = c_0 H + \text{const} \). The general solution is then

\[
u = \text{const} + \sum_n c_n (\tau_n H).
\]

This is formal – the series doesn’t necessarily converge in \( D'(\mathbb{R}) \). Take \( c_n = 0 \) for all but finitely many \( n \).
We are required to compute
\[ \langle \partial_x^2 u - \partial_y^2 u, \varphi \rangle = \iint u(x, y) (\varphi_{xx} - \varphi_{yy}) \, dx \, dy \]
We do the first term first: since \( u = 0 \) when \( x < y \) we have
\[ \iint u(x, y) \varphi_{xx}(x, y) \, dx \, dy = \int \left( \int_{y}^{\infty} \varphi_{xx}(x, y) \, dx \right) \, dy = - \int \varphi_x(x, y) \, dy. \]
Similarly for the second term
\[ -\iint u(x, y) \varphi_{yy}(x, y) \, dx \, dy = -\int \left( \int_{-\infty}^{x} \varphi_{yy}(x, y) \, dy \right) \, dx = -\int \varphi_y(x, x) \, dx. \]
Summing up these contributions and relabelling the dummy variables to \( \tau \) we have
\[ \langle \partial_x^2 u - \partial_y^2 u, \varphi \rangle = -\int \left( \varphi_x(\tau, \tau) + \varphi_y(\tau, \tau) \right) \, d\tau = -\int \frac{d}{d\tau} \varphi(\tau, \tau) \, d\tau = 0 \]
i.e. \( \partial_x^2 u - \partial_y^2 u = 0 \) in \( \mathcal{D}'(\mathbb{R}^2) \). If \( y \) is a time-like coordinate then this corresponds to a “step” wave.

9. Since \( |x|^{2-n} \) is locally integrable on \( \mathbb{R}^n \) it follows that
\[ \lim_{\epsilon \to 0} \int_{B_{\epsilon}} \frac{\varphi}{|x|^{n-2}} \, dx = 0 \]
(this can be checked using polar coordinates with \( dx = r^{n-1} \, dr \, d\sigma_{n-1} \)). To look at the contribution from \( B_{\epsilon}^c \), first choose \( R > 0 \) sufficiently large so that \( \text{supp}(\varphi) \subset B_R \). Now apply Green’s theorem to \( \Omega_{\epsilon} = B_R \setminus B_{\epsilon} \) and observe that \( \Delta(|x|^{2-n}) = 0 \) on \( B_{\epsilon}^c \) for any \( \epsilon > 0 \). Recall Green’s theorem says
\[ \int_{\Omega_{\epsilon}} (u \Delta v - v \Delta u) \, dx = \int_{\partial \Omega_{\epsilon}} \left( \frac{\partial u}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dA \]
where \( dA \) is the surface element on \( \partial \Omega_{\epsilon} \). Setting \( u = |x|^{2-n}, v = \varphi \) and using polar coordinates with \( r = |x| \) we find
\[ \int_{\Omega_{\epsilon}} \frac{\Delta \varphi}{|x|^{n-2}} \, dx = -\int_{r=\epsilon}^{r=\infty} \left( r^{2-n} \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial}{\partial r} (r^{2-n}) \right) r^{n-1} \, d\sigma_{n-1} \]
where \( d\sigma_{n-1} \) is the surface measure on \( S^{n-1} \). We have chosen \( R > 0 \) sufficiently large so that there are no contributions from the outer surface \( \partial B_R \). The limit of the integral on the right is just
\[ \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{\partial \varphi}{\partial r} \bigg|_{r=\epsilon} - (2-n) \varphi \bigg|_{r=\epsilon} \, d\sigma_{n-1} = (n-2) \omega_{n-1} \varphi(0) \]
where \( \omega_{n-1} = \int_{S^{n-1}} d\sigma_{n-1} \). We deduce that \( \Delta(|x|^{2-n}) = c_n \delta_0 \) in \( \mathcal{D}(\mathbb{R}^n) \), where \( c_n = (2-n) \omega_{n-1} \).

10. First few are straightforward, e.g.
\[ \lim_{k \to \infty} \langle f_k, \varphi \rangle = \lim_{k \to \infty} \int \frac{k \varphi(x)}{1 + k^2 \pi^2} \, dx = \lim_{k \to \infty} \int \frac{\varphi(x/k)}{1 + x^2} \, dx = \frac{\varphi(0)}{1} \]
where in the final step we used the dominated convergence theorem to interchange the limit and the integral. Using the same substitution trick we find (a) \( f_k \to \sqrt{\pi} \delta_0 \). Integrating by parts we find (b) \( f_k \to 0 \). For (c) proceed as follows: given \( \varphi \in \mathcal{D}(\mathbb{R}) \) fix \( R > 0 \) so that \( \text{supp}(\varphi) \subset [-R, R] \). Then
\[ \langle f_k, \varphi \rangle = \int_{-R}^{R} \frac{\sin(kx)}{\pi x} \left[ \varphi(x) - \varphi(0) \right] \, dx + \varphi(0) \int_{-R}^{R} \frac{\sin(kx)}{\pi x} \, dx \]
Using integration by parts on the first term we see
\[ \int_{-R}^{R} \frac{\sin(kx)}{\pi x} \left[ \frac{\varphi(x) - \varphi(0)}{\pi x} \right] \, dx = O \left( \frac{1}{k} \right) \]
where the implied constant depends on \( \varphi \). By changing variables in the second integral we deduce

\[
\lim_{k \to \infty} \langle f_k, \varphi \rangle = \varphi(0) \lim_{k \to \infty} \int_{-kR}^{kR} \frac{\sin x}{\pi x} \, dx = \varphi(0)
\]

so that \( f_k \to \delta_0 \) in \( \mathcal{D}'(\mathbb{R}) \).

12. Any \( \varphi \in \mathcal{D}(-1,1) \) has an absolutely convergent Fourier series

\[
\varphi(x) = \sum_n e^{imx} \hat{\varphi}_n, \quad \hat{\varphi}_n = \frac{1}{2} \int_{-1}^{1} e^{-imx} \varphi(x) \, dx.
\]

We have

\[
\langle u_k, \varphi \rangle = \sum_{|m| \leq k} \frac{1}{2} \int_{-1}^{1} e^{-imx} \varphi(x) \, dx = \sum_{|m| \leq k} \hat{\varphi}_n
\]

From this we deduce

\[
\lim_{k \to \infty} \langle u_k, \varphi \rangle = \sum_n \hat{\varphi}_n = \varphi(0)
\]

i.e. \( u_k \to \delta_0 \) in \( \mathcal{D}'(-1,1) \).

13. For a given \( \varphi \in \mathcal{D}(\mathbb{R}) \) fix \( R > 0 \) so that \( \text{supp}(\varphi) \subset [-R, R] \). Then

\[
\int_{|x| \geq R} \frac{\varphi(x)}{x} \, dx = \left[ \int_{-R}^{0} + \int_{0}^{R} \right] \frac{\varphi(x)}{x} \, dx = \int_{R}^{0} \frac{\varphi(x) - \varphi(-x)}{x} \, dx
\]

It follows that

\[
\left\langle \text{p.v.} \left( \frac{1}{x} \right), \varphi \right\rangle = \int_{0}^{R} \frac{\varphi(x) - \varphi(-x)}{x} \, dx
\]

For each \( [-R, R] \subset \mathbb{R} \) with \( \text{supp}(\varphi) \subset [-R, R] \) we have

\[
\left| \left\langle \text{p.v.} \left( \frac{1}{x} \right), \varphi \right\rangle \right| \leq \int_{0}^{R} \left| \frac{\varphi(x) - \varphi(-x)}{x} \right| \, dx = \int_{0}^{R} \frac{1}{x} \int_{-x}^{x} \left| \varphi'(y) \right| \, dy \, dx \leq \int_{0}^{R} \int_{-x}^{x} \varphi'(x) \, dx \, dx
\]

hence

\[
\left| \left\langle \text{p.v.} \left( \frac{1}{x} \right), \varphi \right\rangle \right| \leq C_R \sup |\varphi'| \leq C_R \sum_{|n| \leq 1} \sup |\partial^n \varphi|.
\]

So \( \text{p.v.}(1/x) \in \mathcal{D}'(\mathbb{R}) \) and \( \text{ord}(\text{p.v.}(1/x)) \leq 1 \). If \( \text{ord}(\text{p.v.}(1/x)) = 0 \) then \( \left| \langle \text{p.v.}(1/x), \varphi \rangle \right| \leq C_K \| \varphi \|_{\infty} \) for all \( \varphi \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp} \varphi \subset K \). But this can’t be true: consider \( 0 \leq \varphi_n \leq 1 \) with \( \text{supp} \varphi_n \subset [0,2] \) and \( \varphi_n = 1 \) on \([1/n,1]\). Then since \( \varphi_n(0) = 0 \) for each \( n \) (by continuity) we have

\[
\left\langle \text{p.v.} \left( \frac{1}{x} \right), \varphi_n \right\rangle = \int_{0}^{1} \frac{\varphi_n(x)}{x} \, dx \geq \int_{1/n}^{1} \frac{\varphi_n(x)}{x} \, dx = \log n.
\]

So \( \langle \text{p.v.}(1/x), \varphi_n \rangle \to \infty \) but \( \| \varphi_n \|_{\infty} = 1 \) and \( \text{supp} \varphi_n \subset K = [0,2] \), so \( \text{ord}(\text{p.v.}(1/x)) > 0 \), and by previous it must be \( 1 \). A note for intuition: we picked a collection of test functions that were bounded, but whose derivatives necessarily got bigger and bigger near \( x = 0 \). For the next part

\[
\int_{-R}^{R} \frac{\varphi(x)}{x} \, dx = \int_{-R}^{R} \varphi(x) - \varphi(0) \, dx + \varphi(0) \int_{-R}^{R} \frac{dx}{x} = \int_{-R}^{R} \frac{\varphi(x)}{x} \, dx + i \varphi(0) \int_{-R}^{R} \frac{\epsilon \, dx}{x^2 + \epsilon^2} + \varphi(0) \int_{-R}^{R} \frac{x}{x^2 + \epsilon^2} \, dx 
\]

\[
= \int_{-R}^{R} \frac{\varphi(x)}{x} \, dx + i \varphi(0) \int_{-R/\epsilon}^{R/\epsilon} \frac{dx}{x^2 + 1}
\]
where we discarded one integral since the integrand was odd. Taking the limit we get

$$\lim_{\epsilon \to 0} \left< \frac{1}{x - i\epsilon}, \varphi \right> = \int_{-R}^{R} \frac{\varphi(x) - \varphi(0)}{x} \, dx + i\pi \varphi(0)$$

Since the integrand in the first term is integrable, we can write it as

$$\int_{-R}^{R} \frac{\varphi(x) - \varphi(0)}{x} \, dx = \lim_{\delta \to 0} \int_{\delta < |x| < R} \frac{\varphi(x) - \varphi(0)}{x} \, dx = \lim_{\delta \to 0} \int_{\delta < |x| < R} \frac{\varphi(x)}{x} \, dx - \lim_{\delta \to 0} \int_{\delta < |x| < R} \frac{\varphi(0)}{x} \, dx$$

The second of these vanishes and we deduce

$$\lim_{\epsilon \to 0} \left< \frac{1}{x - i\epsilon}, \varphi \right> = \left< \text{p.v.} \left( \frac{1}{x} \right), \varphi \right> + \{i\pi \delta_0, \varphi\}$$

which yields the desired result.

14. We have

$$\langle (\log |x|)', \varphi \rangle = -\int \log |x| \varphi'(x) \, dx$$

$$= -\lim_{\epsilon \to 0} \int_{|x| > \epsilon} \log |x| \varphi'(x) \, dx$$

$$= \lim_{\epsilon \to 0} \left[ \log(\epsilon) \varphi(-\epsilon) - \log(\epsilon) \varphi(\epsilon) + \int_{|x| > \epsilon} \frac{\varphi(x)}{x} \, dx \right]$$

$$= \left< \text{p.v.} \left( \frac{1}{x} \right), \varphi \right>$$

where we used the fact \( \varphi(\epsilon) - \varphi(-\epsilon) = \mathcal{O}(\epsilon) \) and \( \epsilon \log(\epsilon) = o(1) \).

15. First we split the integral into neighbourhoods of the zeros of \( f \) and their complement

$$\langle \Delta \log |f|, \varphi \rangle = \int_X \log |f| \Delta \varphi \, dx = \sum_i \int_{B_\epsilon(z_i)} \log |f| \Delta \varphi \, dx + \int_{X \setminus \cup B_\epsilon(z_i)} \log |f| \Delta \varphi \, dx.$$ 

The first collection of integrals can be made as small as we please by making \( \epsilon \) small. Indeed, in a neighbourhood of any \( z_i \) we can write \( f(z) = (z - z_i)^m g(z) \) for some analytic \( g \) with \( g(z_i) \neq 0 \) so

$$\log |f| = m_i \log |z - z_i| + \log |g|$$

in a neighbourhood of \( z_i \). So \( \log |f| \) is clearly locally integrable. By Greens theorem

$$\int_{X \setminus \cup B_\epsilon(z_i)} \left( \varphi \Delta \log |f| - \log |f| \Delta \varphi \right) \, dx = \sum_i \int_{\partial B_\epsilon(z_i)} \left( \varphi \frac{\partial}{\partial n} \log |f| - \log |f| \frac{\partial \varphi}{\partial n} \right) \, ds$$

where \( s \) is the arc-length. By translation invariance, it is enough to understand the problem about \( z_i = 0 \). Since \( \log |f| = \text{Re} \log(f) \) we know that \( \log |f| \) is harmonic in \( X \setminus \cup B_\epsilon(z_i) \) so we have

$$- \int_{X \setminus \cup B_\epsilon(z_i)} \log |f| \Delta \varphi \, dx = -\sum_i \int_{r=\epsilon}^{r=\infty} \left( \varphi \frac{\partial}{\partial r} (m_i \log r + \log |g|) - (m_i \log r + \log |g|) \frac{\partial \varphi}{\partial r} \right) r \, dr$$

$$= -m_i \int_0^{2\pi} \varphi(\epsilon, \theta) \, d\theta + \mathcal{O}(\epsilon)$$

where the minus sign comes from the fact the normal is pointing out of \( X \setminus \cup B_\epsilon(z_i) \), i.e. into \( B_\epsilon(z_i) \) in this instance. Taking the limit \( \epsilon \to 0 \) we deduce

$$\langle \Delta \log |f|, \varphi \rangle = 2\pi \sum_i m_i \langle \delta_{z_i}, \varphi \rangle$$

hence the desired result.
16. For $\varphi \in \mathcal{E}(\mathbb{R}^3)$ the divergence theorem gives

$$\left\langle - \sum x_i \frac{\partial u}{\partial x_i}, \varphi \right\rangle = \sum_i \int_{B_1} \frac{\partial}{\partial x_i} (x_i \varphi) \, dx = \int_{B_1} \nabla \cdot (x \varphi) \, dx = \int_{S^2} (x \varphi) \cdot x \, d\sigma_2 = \int_{S^2} \varphi \, d\sigma_2$$

since $x \cdot x = 1$ for $x \in S^2$. 