1. Recall that $u * v$ is defined by $(u * v) \varphi(x) := u * (v \varphi)(x)$ for $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and let us suppose $v$ has compact support. Note that $(u * v)(x, y) = u * (v \varphi)(0) = u * (v \varphi)(0)$. Suppose $\{\varphi_m\}_{m \geq 0}$ is sequence that tends to zero in $\mathcal{D}(\mathbb{R}^n)$, then the same can be said for the sequence $\psi_m := v * \varphi_m$. Indeed, we have $\sup \psi_m \leq \sup v + \sup \varphi_m \leq \sup v + \sup K$ for some compact $K$ for which $\sup \varphi_m \subset K$ for each $m$. And also $\partial^\alpha \psi_m \to 0$ locally uniformly for each multi-index $\alpha$, since

$$|\partial^\alpha \psi_m(x)| = |v * \partial^\alpha \varphi_m(x)| \lesssim \sum_{|\beta| \leq N} \sup \partial^\beta \varphi_m(x - y)|$$

for some compact $K'$ and some fixed $N$, and since the right hand side tends to zero uniformly in $x$ we conclude $\psi_m \to 0$ in $\mathcal{D}(\mathbb{R}^n)$. So if $\varphi_m \to 0$ in $\mathcal{D}(\mathbb{R}^n)$ then $\langle u * v, \hat{\varphi}_m \rangle = u * \psi_m(0) = \langle u, \psi_m \rangle \to 0$, so $u * v \in \mathcal{D}(\mathbb{R}^n)$ by the sequential continuity definition. Uniqueness is straightforward.

2. We can write the Fourier series as

$$\varphi(x) = \frac{1}{2\pi} \sum_{\lambda \in \Lambda} (\pi \epsilon) e^{i \lambda \epsilon} \hat{\varphi}(\lambda)$$

where $\Lambda = \{\lambda \in \mathbb{R} : \lambda/\pi \epsilon \in \mathbb{Z}\}$. Since $\mathbb{R}$ is the disjoint union of intervals of length $(\pi \epsilon)$ centred about the points in $\Lambda$, it follows that the series on the right is the Riemann sum for the integral

$$\frac{1}{2\pi} \int e^{i \lambda \epsilon} \hat{\varphi}(\lambda) \, d\lambda.$$

Taking the limit gives the required result.

3. Consider the function $F(x) = \sum_m \varphi(m + x)$. Then $F$ is a periodic function with period 1, so has a Fourier series $F(x) = \sum_n e^{2\pi i nx} \hat{F}_n$ where

$$\hat{F}_n = \int_0^1 e^{-2\pi i nx} F(x) \, dx = \sum_m \int_0^1 e^{-2\pi i nx} \varphi(m + x) \, dx$$

(uniform convergence justifies interchange of integral and sum). The right hand side can be written

$$\sum_m \int_m^{m+1} e^{-2\pi i nx} \varphi(x) \, dx = \int e^{-2\pi i nx} \varphi(x) \, dx = \hat{\varphi}(2\pi n).$$

Hence $\sum_m \varphi(m + x) = \sum_n e^{2\pi i nx} \hat{\varphi}(2\pi n)$ and the result follows by taking $x = 0$.

4. Just use $\langle \lambda \rangle^s \leq \langle \lambda \rangle^t$ if $s < t$.

5. To prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi \, dx = 1$ and set $\varphi_\epsilon(x) = \epsilon^{-n} \varphi(x/\epsilon)$. Then for $u \in L^2(\mathbb{R}^n)$ the function $u_\epsilon(x) = u * \varphi_\epsilon(x)$ is well defined (by Cauchy-Schwarz inequality) and it is straightforward to prove that $u_\epsilon \in \mathcal{S}(\mathbb{R}^n)$ for each $\epsilon > 0$. Note that

$$|u_\epsilon(x) - u(x)|^2 = \left| \int [u(x, y) - u(x)] \varphi_\epsilon(y) \, dy \right|^2 \leq \int |\varphi_\epsilon(y)| \, dy \int |\varphi_\epsilon(y)||u(x, y) - u(x)|^2 \, dy$$

using Cauchy-Schwarz. Hence

$$|u_\epsilon(x) - u(x)|^2 \leq \int |\varphi(y)||u(x - \epsilon y) - u(x)|^2 \, dy.$$
Using Fubini’s theorem we deduce

\[ \|u - u\|_{L^2}^2 \lesssim \int \|\tau_y u - u\|_{L^2}^2 |\varphi(y)| \, dy. \]

Note that Parseval’s theorem gives

\[ \|\tau_y u - u\|_{L^2}^2 \lesssim \|(\tau_y u) - \tilde{u}\|_{L^2}^2 = \int |(e^{-iy\cdot\lambda} - 1)\tilde{u}(\lambda)|^2 \, d\lambda \to 0 \]

by dominated convergence. So, again by dominated convergence, we find \( \|u - u\|_{L^2} \to 0 \), i.e. \( u \to u \) in \( L^2(\mathbb{R}^n) \), so \( S(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \). Now for \( u \in H^s(\mathbb{R}^n) \) we know that \( (\lambda)^s \tilde{u} \in L^2(\mathbb{R}^n) \). So there exists a sequence of Schwartz functions \( \{\psi_k\}_{k \geq 1} \) such that \( \psi_k \to (\lambda)^s \tilde{u} \) in \( L^2(\mathbb{R}^n) \). Consider the sequence

\[ \varphi_k = \mathcal{F}^{-1}((\lambda)^{-s} \psi_k) \quad k = 1, 2, 3, \ldots \]

This sequence certainly is Schwartz and from the definitions we have

\[ \|u - \varphi_k\|_{H^s} = \|\langle \lambda \rangle^s (\tilde{u} - \langle \lambda \rangle^{-s} \psi_k)\|_{L^2} = \|\langle \lambda \rangle^s \tilde{u} - \psi_k\|_{L^2} \to 0 \]

so we’re done.

6. We have

\[ \langle (D^\alpha u), \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \tilde{\varphi} \rangle = \langle u, (x^\alpha \varphi)' \rangle = \langle \lambda^\alpha \tilde{u}, \varphi \rangle. \]

The second is similar.

7. Take \( \varphi \in D(\mathbb{R}^n) \). Then for \( u \in \mathcal{E}'(\mathbb{R}^n) \) we have

\[ \langle \tilde{u}, \varphi \rangle = \langle u, \varphi \rangle = \langle u(x), \int e^{-i\lambda \cdot x} \varphi(\lambda) \, d\lambda \rangle = \int \langle u(x), e^{-i\lambda \cdot x} \varphi(\lambda) \rangle \, d\lambda \]

where the interchange follows from the fact that \( \varphi \) has compact support and the Riemann sum converges uniformly to the integral so that the interchange is justified. Now \( D(\mathbb{R}^n) \) is dense \(^3\) in \( S(\mathbb{R}^n) \) so for arbitrary \( \varphi \in S(\mathbb{R}^n) \) we can find \( \varphi_k \in D(\mathbb{R}^n) \) \( (k = 1, 2, \ldots) \) with \( \varphi_k \to \varphi \) in \( S(\mathbb{R}^n) \). Hence

\[ \langle u, \varphi \rangle = \lim_{k \to \infty} \int \langle u(x), e^{-i\lambda \cdot x} \varphi_k(\lambda) \rangle \, d\lambda = \int \langle u(x), e^{-i\lambda \cdot x} \varphi(\lambda) \rangle \, d\lambda \]

where in the final line we used the dominated convergence theorem. It follows from the semi-norm estimates defining \( \mathcal{E}'(\mathbb{R}^n) \) that \( |u(\lambda)| \lesssim |\lambda|^N \) for some \( N \), so immediately \( u \in H^s(\mathbb{R}^n) \) for some \( s \in \mathbb{R} \).

8. Let us assume \( u_2 \) has compact support (if not, relabel \( u_1 \) by \( u_2 \) and use \( u_1 \ast u_2 = u_2 \ast u_1 \)). First we note that the product is well defined since \( \tilde{u}_2 \in C^\infty(\mathbb{R}^n) \) and has polynomial growth by the previous question. For all \( \varphi \in D(\mathbb{R}^n) \) we have by definition of the convolution and \( \langle u, \varphi \rangle = u \ast \tilde{\varphi}(0) \) that

\[ \langle u_1 \ast u_2, \varphi \rangle = u_1 \ast (u_2 \ast \tilde{\varphi})(0) = \langle u_1, (u_2 \ast \tilde{\varphi})' \rangle = \langle u_1, \tilde{u}_2 \ast \varphi \rangle. \]

Note that \( \tilde{u}_2 \ast \varphi \in D(\mathbb{R}^n) \), since \( u_2 \) has compact support. So for \( \varphi \in D(\mathbb{R}^n) \) we certainly have

\[ \langle (u_1 \ast u_2)', \varphi \rangle = \langle u_1 \ast u_2, \varphi \rangle = \langle u_1, \tilde{u}_2 \ast \varphi \rangle. \]

Now if \( \varphi \in D(\mathbb{R}^n) \) then \( \tilde{u}_2 \ast \tilde{\varphi} \) is the Fourier transform of \( \tilde{u}_2 \varphi \), since

\[ \int e^{-i\lambda \cdot x} \tilde{u}_2(\lambda) \varphi(\lambda) \, d\lambda = \int e^{-i\lambda \cdot x} \tilde{u}_2(y), e^{i\lambda \cdot y} \varphi(\lambda) \, d\lambda = \left\langle \tilde{u}_2(y), \int e^{-i\lambda \cdot (x-y)} \varphi(\lambda) \, d\lambda \right\rangle = \tilde{u}_2 \ast \varphi(x). \]

So for \( \varphi \in D(\mathbb{R}^n) \) we certainly have

\[ \langle (u_1 \ast u_2)', \varphi \rangle = \langle u_1, (\tilde{u}_2 \varphi)' \rangle = \langle u_1 \tilde{u}_2, \varphi \rangle \]

and the result follows from the density of \( D(\mathbb{R}^n) \) in \( S(\mathbb{R}^n) \).

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\(^3\)For \( \varphi \in S(\mathbb{R}^n) \) take \( \varphi_k \in D(\mathbb{R}^n) \) \( (k = 1, 2, \ldots) \) with \( \varphi_k(x) = \chi(x/k) \varphi(x) \) where \( \chi \in D(\mathbb{R}^n) \) is equal to one on \( |x| < 1 \) and zero in \( |x| > 2 \). Then \( \varphi_k \to \varphi \) in \( S(\mathbb{R}^n) \).
9. By the dominated convergence theorem
\[
\lim_{\epsilon \to 0} \langle e^{-\epsilon x} H, \varphi \rangle = \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon x} \varphi(x) \, dx = \int_0^\infty \varphi(x) \, dx = \langle H, \varphi \rangle
\]
for all \( \varphi \in S(\mathbb{R}^n) \). So \( e^{-\epsilon x} H \to H \) in \( S'(\mathbb{R}^n) \). We also have
\[
\langle (e^{-\epsilon x} H), \varphi \rangle = \int_0^\infty e^{-\epsilon x} \varphi(x) \, dx = \int \left( \int_0^\infty e^{-(i\lambda + \epsilon)x} \, dx \right) \varphi(\lambda) \, d\lambda = \int \frac{\varphi(\lambda)}{i\lambda + \epsilon} \, d\lambda
\]
where we used Fubini’s theorem to swap the order of integration. Since the Fourier transform is a continuous isomorphism on \( S'(\mathbb{R}^n) \) we deduce
\[
\hat{H} = \lim_{\epsilon \to 0} (e^{-\epsilon x} H) = \lim_{\epsilon \to 0} \frac{-i}{x - i\epsilon} = \pi \delta_0 - \text{p.v.} \left( \frac{1}{x} \right)
\]
in \( S'(\mathbb{R}^n) \).

10. Using the suggested substitution the integral for the Fourier transform is
\[
\int e^{-i\lambda x} u(x) \, dx = \frac{1}{2} \int \left[ e^{-i\lambda x} u(x) + e^{-i\lambda(x + \pi/\lambda)} u \left( x + \frac{\pi}{\lambda} \right) \right] \, dx = \frac{1}{2} \int e^{-i\lambda x} \left[ u(x) - u \left( x + \frac{\pi}{\lambda} \right) \right] \, dx.
\]
Since \( u \in L^1(\mathbb{R}) \) we can choose \( R = R(\epsilon) \) sufficiently large so that
\[
\int_{|x| > R} \left| u(x) - u \left( x + \frac{\pi}{\lambda} \right) \right| \, dx < \frac{\epsilon}{4}
\]
for any \( \epsilon > 0 \). If \( u \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \) we can use the dominated convergence theorem to choose \( \lambda = \lambda(\epsilon) \) sufficiently large so that
\[
\int_{|x| < R} e^{-i\lambda x} \left[ u(x) - u \left( x + \frac{\pi}{\lambda} \right) \right] \, dx < \frac{\epsilon}{4}.
\]
So given \( \epsilon > 0 \) we know we can make \( |\lambda| \) sufficiently large so that \( |\hat{u}(\lambda)| < \epsilon/2 \) if \( u \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \). But these functions are dense in \( L^1(\mathbb{R}) \), so for \( u \in L^1(\mathbb{R}) \) first choose \( u_\epsilon \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \) such that \( \|u - u_\epsilon\|_{L^1} < \epsilon/2 \). Then by the triangle inequality
\[
|\hat{u}(\lambda)| \leq \left\| e^{-i\lambda x} (u(x) - u_\epsilon(x)) \right\| + |\hat{u}_\epsilon(\lambda)| \leq \|u - u_\epsilon\|_{L^1} + |\hat{u}_\epsilon(\lambda)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
for \( |\lambda| \) sufficiently large.

11. If you haven’t seen any functional analysis or measure theory before, then you can assume that if \( \hat{u} \in L^2(\mathbb{R}^n) \) then \( u \) can be realised by a bona fide function (see the appendix). We have
\[
\langle \lambda \rangle^{2m} = (1 + |\lambda|^2)^m \leq 2^m (1 + |\lambda|^{2m}) \leq \sum_{|\alpha| \leq m} |\lambda^\alpha|^2.
\]
So we have
\[
\int \langle \lambda \rangle^{2m} |\hat{u}|^2 \, d\lambda \leq \sum_{|\alpha| \leq m} \int |\lambda^\alpha \hat{u}|^2 \, d\lambda = \sum_{|\alpha| \leq m} \int |(D^\alpha u)(\lambda)|^2 \, d\lambda = \frac{1}{(2\pi)^n} \sum_{|\alpha| \leq m} |D^\alpha u|^2 \, dx
\]
The other direction is similar. This shows that the norms are equivalent.

12. It is clear that \( \mathcal{O}(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \) and in fact multiplication by elements of \( \mathcal{O}(\mathbb{R}^n) \) is a continuous map on \( S'(\mathbb{R}^n) \). By definition of the Fourier transform and the dominated convergence theorem
\[
\langle \hat{u}, \psi \rangle = \int u(x) \hat{\psi}(x) \, dx = \lim_{\epsilon \to 0} \int \varphi(\epsilon x) u(x) \left( \int e^{-i\lambda \cdot x} \varphi(\lambda) \, d\lambda \right) = \lim_{\epsilon \to 0} \int \left( \int e^{-i\lambda \cdot x} \varphi(\epsilon x) u(x) \, dx \right) \psi(\lambda) \, d\lambda
\]
and the result follows.
13. By dominated convergence and Fubini’s theorem

\[ \langle \hat{u}, \varphi \rangle = \lim_{R \to \infty} \int_{-R}^{R} \frac{x}{1 + x^2} \left[ \int e^{-i\lambda x} \varphi(\lambda) \, d\lambda \right] \, dx = \lim_{R \to \infty} \int \left[ \int_{-R}^{R} \frac{xe^{-i\lambda x}}{1 + x^2} \, dx \right] \varphi(\lambda) \, d\lambda. \]

For \( \lambda = 0 \) the Fourier transform vanishes since the integrand is odd. For \( \lambda > 0 \) the integrand decays exponentially in the lower half of the complex \( x \)-plane. Closing up our contour there using Jordan’s lemma we pick up (minus) the residues from the lower half plane (assuming \( R > 1 \))

\[ \int_{-R}^{R} \frac{xe^{-i\lambda x}}{1 + x^2} \, dx = -2\pi i \left[ \frac{1}{2} e^{-\lambda} + O \left( \frac{1}{R} \right) \right] \]

uniformly in \( \lambda \in \mathbb{R} \). Closing the the upper half plane for \( \lambda < 0 \) gives

\[ \int_{-R}^{R} \frac{xe^{-i\lambda x}}{1 + x^2} \, dx = 2\pi i \left[ \frac{1}{2} e^{\lambda} + O \left( \frac{1}{R} \right) \right]. \]

We deduce that \( \hat{u}(\lambda) = -\pi i e^{-|\lambda|} \text{sgn}(\lambda) \), so \( u \in H^s(\mathbb{R}) \) for each \( s \in \mathbb{R} \).

14. We know that \( (D^\alpha \delta_0)(\lambda) = \lambda^\alpha \). Then

\[ \|D^\alpha \delta_0\|_{H^s}^2 = \int \langle \lambda \rangle^s |\lambda^\alpha|^2 \, d\lambda. \]

Using spherical polars we find

\[ \|D^\alpha \delta_0\|_{H^s}^2 = \int_{S^{n-1}} |f_{\alpha,s}(\theta)| \, d\sigma_{n-1} \int_0^\infty (1 + r^2)^{s+|\alpha|+n-1} \, dr \]

for some function on \( f_{\alpha,s} \) on \( S^{n-1} \). The final integrand is \( O(r^{2s+2|\alpha|+n-1}) \) for large \( r \), so the integral converges if and only if \( s < -|\alpha| - \frac{1}{2} n \).

15. By definition of the Fourier transform and Fubini’s theorem

\[ \langle \hat{\mu}_\Gamma, \varphi \rangle = \int_{\Gamma} \left[ \int e^{-i\lambda \cdot x} \varphi(\lambda) \, d\lambda \right] d\mu = \int \left[ \int_{\Gamma} e^{-i\lambda \cdot x} d\mu \right] \varphi(\lambda) \, d\lambda, \]

so the first result follows. For the next part, write \( \pi : \mathbb{R}^n \to \Gamma \) for the projection of a vector \( \lambda \in \mathbb{R}^n \) into \( \Gamma \). Then we can write \( \lambda = \pi(\lambda) + (\lambda \cdot n)n \) and

\[ \hat{\mu}(\lambda) = \int_{\Gamma} e^{-i\lambda \cdot x} d\mu(x) = \int_{\Gamma} e^{-i\pi(\lambda) \cdot x} d\mu(x). \]

So the integrand has rapid decay in the \( \Gamma \) plane but no decay in the direction of the normal \( n \). It follows that \( \mu_\Gamma \in H^s(\mathbb{R}^n) \) if \( s < -\frac{1}{2} \).

16. Taking Fourier transforms gives \( \hat{u} = (|\lambda|^4 + 1)^{-1} \hat{f} \). The rest is now routine.

17. For (a) use \( \text{sgn}(x) = H(x) - H(-x) \) so that \( (\text{sgn})^\prime(\lambda) = -2i \text{p.v.}(1/\lambda) \). For (b) and (c) use the fact that \( \lambda^\alpha \hat{u} = (D^\alpha \hat{u})^\prime \). With this in mind you should compute the Fourier transforms of

\[ -i \frac{d}{dx}\text{arctan}(x) = \frac{-i}{1 + x^2}, \quad -i \frac{d^2}{dx^2} (x \log |x| - x) = -\text{p.v.} \left( \frac{1}{x} \right). \]

The Fourier transforms are \( -i\pi e^{-|\lambda|} \) and \( \pi \text{sgn}(\lambda) \). We are left with finding solutions to the equations

(b) \( \lambda \hat{u} = -i\pi e^{-|\lambda|} \), \quad (c) \( \lambda^2 \hat{u} = i\pi \text{sgn}(\lambda) \).
in $\mathcal{S}'(\mathbb{R})$. So we need to find some particular solutions, since the homogeneous problems are solved by Dirac measures and derivatives thereof in the second case (recall example sheet 1). For part (b), it is tempting to “multiply” both sides by p.v.$(1/\lambda)$ and hope for the best – let’s see if it works:

$$\left\langle -i\pi\lambda e^{-|\lambda|} \text{p.v.} \left(\frac{1}{\lambda}\right), \varphi \right\rangle = -i\pi \left\langle e^{-|\lambda|} \text{p.v.} \left(\frac{1}{\lambda}\right), \lambda \varphi \right\rangle = -i\pi \lim_{\epsilon \to 0} \int_{|z| > \epsilon} \frac{e^{-|\lambda|} \lambda \varphi(\lambda)}{\lambda} \, d\lambda = \left\langle -i\pi e^{-|\lambda|}, \varphi \right\rangle$$

which looks good! So the solution for this part is

$$(b) \quad \hat{u}(\lambda) = -i\pi e^{-|\lambda|} \text{p.v.} \left(\frac{1}{\lambda}\right) + c_0 \delta_0$$

for some constant $c_0$. For part (c) consider the distribution p.f.$(1/x^2)$ defined by

$$\left\langle \text{p.f.} \left(\frac{1}{x^2}\right), \varphi \right\rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{\varphi(x) - \varphi(0)}{x^2} \, dx,$$

This certainly defines a tempered distribution (check). I leave it as an exercise for the keen to tweak this distribution so that the answer for this part is

$$(c) \quad \hat{u}(\lambda) = [\star] + c_1 \delta_0 + c_2 \delta'_0$$

for some constants $c_1, c_2$ and some tempered distribution $[\star]$. The super-duper keen should compute the constants $\{c_n\}_{n=0}^2$. For the final one, by dominated convergence and Fubini’s theorem we have

$$\langle \hat{u}, \varphi \rangle = \lim_{\epsilon \to 0} \int \int e^{-(\epsilon - i\omega)x^2 - i\lambda x} \, dx \varphi(\lambda) \, d\lambda$$

Set $z = \epsilon - i\omega$. Then by completing the square and Cauchy’s theorem we have

$$\int e^{-(\epsilon - i\omega)x^2 - i\lambda x} \, dx = e^{-\lambda^2/4\epsilon} \int e^{-x(z + i\lambda/2\epsilon)^2} \, dx = e^{-\lambda^2/4\epsilon} \int e^{-x^2} \, dx = 2e^{-\lambda^2/4\epsilon} \int_0^\infty e^{-x^2} \, dx.$$

For the final integral make the substitution $y = z^{1/2} x$, where we have chosen the square root with positive real part, so that

$$\int e^{-(\epsilon - i\omega)x^2 - i\lambda x} \, dx = \frac{2e^{-\lambda^2/4\epsilon}}{z^{1/2}} \int_\gamma(z) e^{-x^2} \, dx$$

where $\gamma(z)$ is the ray in the complex $x$-plane aligned with $\text{arg}(z^{1/2})$. Using Cauchy’s theorem again we can rotate this contour onto the real positive axis. The final result is

$$\int e^{-(\epsilon - i\omega)x^2 - i\lambda x} \, dx = \frac{\sqrt{\pi} e^{-\lambda^2/2z}}{z^{1/2}}.$$

Now as $\epsilon \downarrow 0$ we have $z^{1/2} \to |\omega|^{1/2} e^{-i\text{sgn}(\omega)\pi/4}$ (draw a picture). Hence

$$\hat{u}(\lambda) = \left(\frac{\pi}{|\omega|}\right)^{1/2} \exp \left(\frac{\pi}{4} \text{sgn} \omega - i\lambda^2/4\omega\right).$$

18. Recall that $E(z, \bar{z}) = 1/\pi z$ is the fundamental solution for the Cauchy-Riemann operator $\partial/\partial \bar{z}$. Taking Fourier transforms of

$$\frac{1}{2\pi} \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \frac{1}{x_1 + ix_2} = \delta_0$$

gives

$$(i\lambda_1 - \lambda_2) \mathcal{F} \left(\frac{1}{x_1 + ix_2}\right) (\lambda_1, \lambda_2) = 2\pi$$

And the result follows.

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2This is a natural extension of p.v.$(1/x)$. The p.f. stands for pseudo-function and these type of distributions arise a lot in quantum field theory where one needs to regularize some seemingly divergent integrals. Fancy sounding physics like infra-red and ultra-violet divergences are related to these distributions (see Advanced Quantum Field Theory course).
19. The information in the question tells us that
\[ f(x) - \frac{x}{1 + x^2} \in L^1(\mathbb{R}) \]
Using the fact that the Fourier transform of an \( L^1(\mathbb{R}) \) function is continuous and the result from question 13 we find
\[ \hat{f}(\lambda) + \pi i e^{-|\lambda|} \text{sgn}(\lambda) \in C(\mathbb{R}). \]
We deduce the jump relation \(-\hat{f}'^+ = i\pi - (-i\pi) = 2\pi i\).

20. The Fourier-Laplace transform of \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) is simply
\[ \hat{\varphi}(z) = \int e^{-iz \cdot x} \varphi(x) \, dx, \quad z \in \mathbb{R}^n. \]
This function is continuous in \( z \), it satisfies the Cauchy-Riemann equations in each \( z_i \) so by the Cauchy-Goursat theorem this function is entire (Hartogs’ theorem tells us that complex analytic in each \( z_i \) implies complex analytic in \( z \)). The estimates follow by noting that
\[ e^{\alpha} \hat{\varphi}(z) = \int (-D_x)^\alpha e^{-iz \cdot x} \varphi(x) \, dx = \int e^{-iz \cdot x} D_x^\alpha \varphi(x) \, dx \]
and the fact \( |e^{-iz \cdot x}| = e^{-z \cdot \text{Im}(z)} \leq e^{\|z\| \cdot |z|} \) and using \( \text{supp}(\varphi) \subset B_\delta \). The other direction is the same as the proof of the Paley-Wiener-Schwartz theorem in lectures, but now you have to apply Cauchy’s theorem in each \( z \)-component.

Appendix
Recall that \( H^s(\mathbb{R}^n) \) consists of distributions \( u \in \mathcal{S}'(\mathbb{R}^n) \) for which \( \hat{u}(\lambda) \) is a function and \( \langle \lambda \rangle^s \hat{u}(\lambda) \in L^2(\mathbb{R}^n) \). In question 11 we would like to show that if \( u \in H^m(\mathbb{R}^n) \) for some \( m \in \mathbb{N} \cup \{0\} \) then \( u \) is a bona fide function function for which
\[ \sum_{|\alpha| \leq m} \int |D^\alpha u|^2 \, dx < \infty. \]
We want to prove the bona-fide function bit, and it suffices to prove this for \( m = 0 \). Using the convolution it is straightforward to show that \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( L^2(\mathbb{R}^n) \), meaning that for each \( \hat{u} \in L^2(\mathbb{R}^n) \) there is a sequence \( \varphi_k \in \mathcal{S}(\mathbb{R}^n) \) such that \( \varphi_k \rightarrow \hat{u} \) in \( L^2(\mathbb{R}^n) \) as \( k \rightarrow \infty \), i.e.
\[ \| \varphi_k - \hat{u} \|_{L^2} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]
We would like to show that the sequence \( \varphi_k \) (i.e. the inverse Fourier transform of \( \hat{\varphi_k} \)) tends to an element of \( L^2(\mathbb{R}^n) \) as \( k \rightarrow \infty \). To do this it suffices to show that \( \varphi_k \) is a Cauchy sequence, since \( L^2(\mathbb{R}^n) \) is complete. For this we use Parseval’s theorem and the triangle inequality
\[ \| \varphi_k - \varphi_l \|_{L^2} = \| \hat{\varphi_k} - \hat{\varphi_l} \|_{L^2} = \| \hat{\varphi_k} - \hat{\varphi_l} + \hat{\varphi_l} \|_{L^2} \leq \| \hat{\varphi_k} - \hat{\varphi_l} \|_{L^2} + \| \hat{\varphi_l} - \hat{\varphi_l} \|_{L^2} \rightarrow 0. \]
So the sequence \( \varphi_k \) is Cauchy in \( L^2(\mathbb{R}^n) \), so \( \lim \varphi_k \in L^2(\mathbb{R}^n) \) and we call this limit \( \hat{u} \). From the distributional point of view we have
\[ \langle u, \hat{\psi} \rangle = \langle \hat{u}, \psi \rangle = \lim_{k \rightarrow \infty} \int \varphi_k \psi \, dx = \lim_{k \rightarrow \infty} \int \varphi_k \hat{\psi} \, dx = \int \hat{u} \hat{\psi} \, dx + \lim_{k \rightarrow \infty} \int (\varphi_k - \hat{u}) \hat{\psi} \, dx \]
By Cauchy-Schwarz we see
\[ \lim_{k \rightarrow \infty} \int |(\varphi_k - \hat{u})| |\hat{\psi}| \, dx \leq \lim_{k \rightarrow \infty} \| \varphi_k - \hat{u} \|_{L^2} \| \hat{\psi} \|_{L^2} = 0 \]
so we can identify the function \( \hat{u} \) with the distribution \( u \). An alternate method of deriving this result is via duality, in particular the Riesz representation theorem. This uses more structure of \( L^2(\mathbb{R}^n) \) – in particular, the fact it is a Hilbert space. If you’re interested in this stuff, it’s a good exercise to obtain the same result we got here via this method. Please don’t worry if this is all new or confusing, this course assumes you don’t know anything about the \( L^p \) spaces and as such, this material is completely non-examinable.