Mathematical Tripos Part II Michaelmas 2015
Integrable Systems, hand out 5: Painlevé transcendents Dr A.C.L. Ashton

Recall that an ODE has the *Painlevé property* if the moveable singularities of its solutions are limited to poles. For example, the ODE

$$\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 = 0, \quad \text{i.e.} \quad w(z) = \frac{1}{\sqrt{2(z-z_0)}}$$

does not have the Painlevé property, whereas

$$\frac{\mathrm{d}w}{\mathrm{d}z} + w^2 = 0, \quad \text{i.e.} \quad w(z) = \frac{1}{z-z_0},$$

does have the Painlevé property. Painlevé classified all the second order ODEs of the form

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = F\left(\frac{\mathrm{d}w}{\mathrm{d}z}, w, z\right)$$

possessing this property, where F is rational function of its arguments. Of these ODEs, all but six were *reducible*, i.e. their solutions can be written in terms of known functions (Sine, Cosine, Jacobi elliptic functions, Bessel functions, Airy functions...). The remaining six *irreducible* equations are called the Painlevé equations. They are, up to simple coordinate transformations

$$\text{(PI)} \quad \frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 6w^2 + z,$$

$$\text{(PII)} \quad \frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha,$$

$$\text{(PIII)} \quad \frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{1}{z}\left(-\frac{\mathrm{d}w}{\mathrm{d}z} + \alpha w^2 + \beta\right) + \gamma w^3 + \frac{\delta}{w},$$

$$\text{(PIV)} \quad \frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3w^3}{2} + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w},$$

$$\text{(PV)} \quad \frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},$$

$$\text{(PVI)} \quad \frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z}$$
$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right).$$

Here $\alpha, \beta, \gamma, \delta$ are constants. The solutions to these equations define new functions, called the Painlevé transcendents.