Mathematical Tripos Part III Michaelmas 2017
Distribution Theory & Applications, Example sheet 3 Answers Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *These answers are* **not** *to be redistributed.*

1. Write the following distributions as sums of derivatives of continuous functions

$$\text{(a) } \delta_0, \qquad \text{(b) } \operatorname{sgn}(x), \qquad \text{(c) p.f.}\left(\frac{1}{x^2}\right).$$

Answer: Use your intuition and prove your guesses are correct. E.g. the first one is $(xH)''$. For the second use $\operatorname{sgn}(x) = H(x) - H(-x)$ with $H = (xH)'$. The third is proportional $(\log|x|)''$.

2. For an Nth order polynomial P in n-variables set $P^{(\alpha)}(\lambda) = \partial^\alpha P(\lambda)$. Show that

$$P(D)[\varphi\psi] = \sum_{|\alpha|\le N} \frac{1}{\alpha!} D^\alpha(\varphi) P^{(\alpha)}(D)(\psi)$$

for all test functions $\psi, \varphi \in \mathcal{D}(\mathbf{R}^n)$.

Answer: Note that it's true for monomials (Leibniz rule), so it's true for polynomials. Alternatively, using the Fourier transform it is sufficient to prove

$$P(\lambda)(\hat\varphi * \hat\psi)(\lambda) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}\left[(D^\alpha\varphi)\hat{} * (P^{(\alpha)}(D)\psi)\hat{}\right](\lambda)$$

Written out in full, the right hand side is

$$\int \left[\sum_{|\alpha|\le N} \frac{1}{\alpha!}(\lambda-\mu)^\alpha P^{(\alpha)}(\mu)\right] \hat\varphi(\lambda-\mu)\hat\psi(\mu)\,\mathrm{d}\mu$$

and by Taylor's theorem the term in parenthesis is just $P(\lambda)$ so we are done.

3. Let $\{\Delta_i\}_{i\ge1}$ and $\{c_i\}_{i\ge1}$ be as in the proof of the Malgrange-Ehrenpreis theorem. Prove that

$$\varphi \mapsto \langle E, \varphi\rangle = \sum_{i=1}^\infty \int_{\Delta_i} \left(\int_{\operatorname{Im}\lambda_n = c_i} \frac{\hat\varphi(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)}\,\mathrm{d}\lambda_n\right)\mathrm{d}\lambda'$$

defines an element of $\mathcal{D}'(\mathbf{R}^n)$.

Answer: Let $\varphi \in \mathcal{D}(\mathbf{R}^n)$ with $\operatorname{supp}\varphi \subset K$. Since $|P(\lambda', \lambda_n)| \gtrsim 1$ uniformly in each integrand we can have

$$|\langle E, \varphi\rangle| \lesssim \sum_{i=1}^\infty \int_{\Delta_i} \int |\hat\varphi(-\lambda', -\lambda_n - \mathrm{i}c_i)|\,\mathrm{d}\lambda_n\,\mathrm{d}\lambda'.$$

Observe that

$$|\lambda^\alpha \hat\varphi(-\lambda', -\lambda_n - \mathrm{i}c_i)| = \left|\int e^{c_i x_n}\varphi(x) D_x^\alpha e^{\mathrm{i}\lambda\cdot x}\,\mathrm{d}x\right| = \left|\int e^{\mathrm{i}\lambda\cdot x} D_x^\alpha[e^{c_i x_n}\varphi(x)]\,\mathrm{d}x\right|$$

Applying the derivatives and using a simple estimate on the integral we see that

$$|\hat\varphi(-\lambda', -\lambda_n - \mathrm{i}c_i)| \lesssim_{c_i, K} \langle\lambda\rangle^{-N} \sum_{|\alpha|\le N} \sup|\partial^\alpha\varphi| \lesssim_K \langle\lambda\rangle^{-N} \sum_{|\alpha|\le N} \sup|\partial^\alpha\varphi|,$$

for any $N \ge 0$. We were able to get rid of the c_i dependency because the c_i are the imaginary parts of complex numbers contained inside a fixed horizontal strip. Hence

$$|\langle E, \varphi\rangle| \lesssim_K \left(\sum_{|\alpha|\le N} \sup|\partial^\alpha\varphi|\right) \sum_{i=1}^\infty \int_{\Delta_i} \int \langle\lambda\rangle^{-N}\,\mathrm{d}\lambda_n\,\mathrm{d}\lambda' \lesssim_K \sum_{|\alpha|\le N} \sup|\partial^\alpha\varphi|$$

where N is chosen sufficiently large so the λ-integral converges. We deduce that $E \in \mathcal{D}'(\mathbf{R}^n)$.

4. Let A be a symmetric $n \times n$ matrix with $\det A \neq 0$ and $\mathsf{Re}\, A \geq 0$ (i.e. $\mathsf{Re}\, A$ is positive semi-definite). Construct, explicitly, a fundamental solution to the operator

$$L = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

[Hint: if A is the identity then L is just the heat operator, so we know the fundamental solution in this case. The fundamental solution for the general case looks similar.]

Answer: Recall that $E(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ for $t > 0$ and zero elsewhere, is a fundamental solution when $A_{ij} = \delta_{ij}$, so we guess a fundamental solution for $t > 0$ of the form

$$E_A(x,t) = c_A t^{-n/2} \exp(-x \cdot Bx/4t)$$

and zero elsewhere, for some $B \in \mathrm{Mat}_n(\mathbf{C})$ and constant c_A, both of which will obviously depend on A. For E_A to be a fundamental solution for L we certainly require LE_A to vanish on $t > 0$. A straightforward computation shows that we require $BAB = B$, i.e. $B = A^{-1}$. We note that $\mathsf{Re}\, B$ is positive semi-definite since $\mathsf{Re}\, A$ is positive definite (exercise). We would like to show that

$$\langle LE_A, \varphi \rangle = \langle E_A, L^* \varphi \rangle = \iint E_A L^* \varphi \, \mathrm{d}x \, \mathrm{d}t = \varphi(0)$$

where $L^* = -\partial_t - A_{ij} \partial_i \partial_j$ is the formal adjoint of L. Just as with the case with $A_{ij} = \delta_{ij}$ we did in lectures, we find that

$$\langle LE_A, \varphi \rangle = \lim_{\epsilon \to 0} \int E_A(x,\epsilon) \varphi(x,\epsilon) \, \mathrm{d}x.$$

Usually we would make the substitution $x = \epsilon^{1/2} x'$ and take the limit, but the difficulty here lies in the fact that the integrand doesn't converge absolutely as $\epsilon \downarrow 0$ if at least one of the σ_i are purely imaginary, so we need to do more work if we want to take the limit inside the integral. If you don't like fiddly limits, just assume A is positive definite and take the limit as usual, else, read on. First we set

$$\varphi(Rx) = P(x) e^{-|x|^2} + \psi(x)$$

where P is a polynomial of order N (the Nth Taylor polynomial of $e^{|x|^2} \varphi$) and ψ vanishes to order $N+1$ at $x = 0$ (i.e. $\partial^\alpha \psi(0) = 0$ when $|\alpha| \leq N+1$). Note that ψ goes to zero very quickly as $|x|$ gets large. Let us concentrate on the first term – we want to compute terms that look like

$$\epsilon^{-n/2} \int e^{-x \cdot Bx/4\epsilon} e^{-|x|^2} x^{2\alpha} \, \mathrm{d}x$$

(we've discarded the parts of P which contain an odd bit since these integrals vanish). Let us first do this integral for $\alpha = 0$. Using the fact[1] that $\int e^{-x \cdot \Lambda x} \, \mathrm{d}x = \pi^{n/2} (\det \Lambda)^{-1/2}$ for positive definite matrices. Letting σ be a diagonal matrix with entries $\sigma_i > 0$ we know that

$$\epsilon^{-n/2} \int e^{-x \cdot Bx/4\epsilon} e^{-x \cdot \sigma x} \, \mathrm{d}x = (4\pi)^{n/2} \det(B + \epsilon\sigma)$$

[1] Assume first that Λ is real and symmetric, so it can be diagonalised by an orthogonal transformation. Using this transformation the resulting integral can be written as the $n-$fold product of one dimensional integrals $\int e^{-\lambda_i x^2} \, \mathrm{d}x$ where each $\lambda_i > 0$ corresponds to the ith eigenvalue of Λ. By Cauchy's theorem this is equivalent to the limit as $R \to \infty$ of

$$2 \int_0^R e^{-\lambda_i x^2} \mathrm{d}x = 2 \int_{\Gamma_R} e^{-\lambda_i z^2} \mathrm{d}z + 2 \int_{C_R} e^{-\lambda_i z^2} \, \mathrm{d}z$$

where Γ_R is the ray in $\mathsf{Re}\, z \geq 0$ with $\arg z = -\frac{1}{2} \arg \lambda_i$ and C_R is the arc which joins this ray back to the real axis. Since $\lambda_i z^2 = |\lambda_i||z|^2$ on Γ_R, this integral converges to $e^{-\mathrm{i} \arg \lambda_i/2} \sqrt{\pi/|\lambda_i|} \equiv \pi^{1/2} \lambda_i^{-1/2}$ in the limit $R \to \infty$, where the square root is positive if λ_i is real and positive. It is straightforward to show that the contribution from C_R vanishes using a variation of Jordan's lemma. We deduce that

$$\int e^{-x \cdot \Lambda x} \, \mathrm{d}x = \pi^{n/2} (\lambda_1 \lambda_2 \cdots \lambda_n)^{-1/2} = \pi^{n/2} (\det \Lambda)^{-1/2}.$$

To extend this result for *complex* positive definite Λ, note that both sides of this result constitute analytic functions of $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ if Λ is positive definite, so a simple analytic continuation argument extends the result.

from the previous result. We simply set $\sigma = I$ so it is relevant for our computation, so in the limit when $\alpha = 0$ we get $(4\pi)^{n/2}(\det B)^{-1/2}$. For $\alpha \neq 0$ we differentiate this result each side with respect to σ_i as many times as we please and then set $\sigma = I$ at the end. We find

$$\epsilon^{-n/2} \int e^{-x\cdot Bx/4\epsilon} e^{-x\cdot\sigma x} x^{2\alpha}\,\mathrm{d}x = \mathcal{O}(\epsilon^{|\alpha|}).$$

So we have that

$$\langle LE_A, \varphi\rangle = \frac{(4\pi)^{n/2} c_A}{(\det B)^{1/2}}\varphi(0) + \lim_{\epsilon\to 0}\left(\epsilon^{-n/2}\int e^{-x\cdot Bx/4\epsilon}\psi(x)\mathrm{d}x\right).$$

The remaining term is easy to deal with using an integration by parts argument. Set $L_i = -\frac{1}{2}A_{ij}\partial/\partial x_j$. Then assuming Einstein summation convention and using the symmetry of B we find

$$L_i e^{-x\cdot Bx/4\epsilon} = (4\epsilon)^{-1} A_{ij}B_{jk}x_k e^{-x\cdot Bx/4\epsilon} = (4\epsilon)^{-1} x_i e^{-x\cdot Bx/4\epsilon}.$$

This allows us to write

$$\int e^{-x\cdot Bx/4\epsilon}\psi(x)\,\mathrm{d}x = 4\epsilon\int \frac{\psi(x)}{x_i} L_i e^{-x\cdot Bx/4\epsilon}\,\mathrm{d}x \quad \text{(no sum)}.$$

We can keep applying this trick many times as we like since $\partial^\alpha\psi(0) = 0$ for $|\alpha| \leq N+1$, where N can be chosen as large as we need. Consequently, the remaining term vanishes in the limit. So we deduce that E_A is indeed a fundamental solution for L with constant $c_A = (4\pi)^{-n/2}(\det A)^{-1/2}$. Shazam[2]!

5. Suppose $L = P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is an Nth order elliptic partial differential operator with constant coefficients. Given $(\lambda, c) \in S^{n-1}\times\mathbf{R}$, show that for R sufficiently large the function

$$u(x;\lambda,c) = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=R}\frac{e^{z(x\cdot\lambda - c)}}{zP(z\lambda)}\,\mathrm{d}z$$

satisfies $L[u] = 1$ with $u = 0$ on the surface $x\cdot\lambda = c$. Here the integral is taken in the complex z-plane around a circle centred at the origin with radius R. How big did R need to be?

Answer: First choose $R > 0$ so that all the roots of the polynomial $p_\lambda(z) = zP(\lambda z)$ are contained inside $|z| < R$. Then the integral is well defined and we have

$$Lu(x) = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=R}\frac{P(z\lambda)e^{z(x\cdot\lambda-c)}}{zP(\lambda z)}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=R}\frac{e^{z(x\cdot\lambda-c)}}{z}\,\mathrm{d}z = 1$$

which follows from the residue theorem. We also have

$$u|_{x\cdot\lambda=c} = \frac{1}{2\pi\mathrm{i}}\oint_{|z|=R}\frac{\mathrm{d}z}{zP(\lambda z)} = 0$$

since we can push the contour of integration out to infinity.

6. Let P be an Nth order polynomial in n-variables and suppose that there exists $C, \delta > 0$ such that $|P^{(\alpha)}(\lambda)| \leq C|\lambda|^{-\delta|\alpha|}|P(\lambda)|$ for all multi-indices α when $|\lambda|$ is sufficiently large. Show

$$P(D)u \in H^s_{\mathrm{loc}}(X) \Rightarrow u \in H^{s+\delta N}_{\mathrm{loc}}(X).$$

Deduce that $P(D)$ is hypoelliptic, i.e. $P(D)u \in C^\infty(X) \Rightarrow u \in C^\infty(X)$.

[2]This is a fairly tough problem, so don't worry if you didn't manage to get it out. It's currently an open (and difficult) problem to show that a similar pointwise limit exists when the test function belongs to some Sobolev space $H^s(\mathbf{R}^n)$ and $A_{ij} = \mathrm{i}\delta_{ij}$. If you're interested in this innocent looking problem, Google the phrase "Pointwise convergence of initial data for Schrödinger operators".

Answer: We need to perturb our bootstrap argument from lectures. Fix $\varphi \in \mathcal{D}(X)$ and choose $\psi_0, \ldots, \psi_M$ with $\operatorname{supp}\psi_{i-1} \subset \operatorname{supp}\psi_i$ and $\psi_{i-1} = 1$ on $\operatorname{supp}\psi_i$ for $i = 1, \ldots, M$. We have $\psi_0 u \in H^t(\mathbf{R}^n)$ for some $t \in \mathbf{R}$.

$$P(D)(\psi_1 u) = \psi_1 P(D)u + \sum_{\alpha \neq 0} D^\alpha \psi_1 P^{(\alpha)}(D)(\psi_0 u)$$

since $\psi_0 = 1$ on $\operatorname{supp}\psi_1$. We deduce that $P(D)(\psi_1 u) \in H^{\tilde{A}_1}(\mathbf{R}^n)$ where $\tilde{A}_1 = \min\{s, t - N + 1\}$. Hence

$$\int \langle\lambda\rangle^{2\tilde{A}_1} |P(\lambda)(\psi_1 u)\hat{}(\lambda)|^2 \, \mathrm{d}\lambda < \infty.$$

Using our (C, δ)-estimate for $P(\lambda)$ we deduce that

$$\int \langle\lambda\rangle^{2(\tilde{A}_1 + |\alpha|\delta)} |P^{(\alpha)}(\lambda)(\psi_1 u)\hat{}(\lambda)|^2 \, \mathrm{d}\lambda < \infty,$$

i.e. $P^{(\alpha)}(D)(\psi_1 u) \in H^{A_1(\alpha)}(\mathbf{R}^n)$ where $A_1(\alpha) = \min\{s + |\alpha|\delta, t - N + 1 + |\alpha|\delta\}$. The next step is

$$P(D)(\psi_2 u) = \psi_2 P(D)u + \sum_{\alpha \neq 0} D^\alpha \psi_2 P^{(\alpha)}(D)(\psi_1 u) \in H^s(\mathbf{R}^n) \oplus \left(\bigoplus_{0 < |\alpha| \leq N} H^{A_1(\alpha)}(\mathbf{R}^n) \right).$$

We deduce $P(D)(\psi_2 u) \in H^{\tilde{A}_2}(\mathbf{R}^n)$, where $\tilde{A}_2 = \min\{s, s + \delta, t - N + 1 + \delta\} = \min\{s, t - N + 1 + \delta\}$. Again using the same argument as we did for $\psi_1 u$ we find that $P^{(\alpha)}(D)(\psi_2 u) \in H^{A_2(\alpha)}(\mathbf{R}^n)$ where

$$A_2(\alpha) = \{s + \delta|\alpha|, t - N + 1 + (1 + |\alpha|)\delta\}.$$

Continuing inductively we deduce that

$$P^{(\alpha)}(D)(\psi_M u) \in H^{A_M(\alpha)}(\mathbf{R}^n), \qquad \text{where } A_M(\alpha) = \min\{s + \delta|\alpha|, t - N + 1 + (M - 1 + |\alpha|)\delta\}.$$

Now choose M sufficiently large so that $s < t - N + 1 + (M - 1)\delta$. Then we find that

$$P^{(\alpha)}(D)(\psi_M u) \in H^{s + \delta|\alpha|}(\mathbf{R}^n).$$

Now choose α with $|\alpha| = N$ so that $P^{(\alpha)}(\lambda) = \text{const} \neq 0$. We deduce $\psi_M u \in H^{s + \delta N}(\mathbf{R}^n)$, and since $\psi_M = 1$ on $\operatorname{supp}\varphi$, we deduce that φu belongs to the same Sobolev space. Since $\varphi \in \mathcal{D}(X)$ was arbitrary it follows that $u \in H^{s + \delta N}_{\text{loc}}(X)$. Hypoellipticity now follows.

7. Give an example of an operator $P(D)$ that is hypoelliptic but not elliptic.

Answer: The Heat operator $\partial_t - \Delta$ is hypoelliptic, but not elliptic (look at its principle symbol). Take the case $P = \partial_t - \partial_x^2$ for simplicity. In this case the symbol is

$$P(\lambda_1, \lambda_2) = \mathrm{i}\lambda_1 + \lambda_2^2.$$

So $|P(\lambda)|^2 = \lambda_1^2 + \lambda_2^4$ for $\lambda \in \mathbf{R}^2$. It's now a straightforward exercise to show that

$$\left| \frac{P^{(\alpha)}(\lambda)}{P(\lambda)} \right|^2 \leq C|\lambda|^{-|\alpha|},$$

so that the previous theorem holds with $\delta = 1/2$.

8. Does there exist a polynomial P of positive degree that satisfies a (C, δ)-estimate given in question 6 with $\delta > 1$? What can you say about $P(D)$ if the polynomial P satisfies the (C, δ)-estimate with $\delta = 1$?

Answer: Let P be an Nth order ($N > 0$) polynomial on $\mathbf{R}^n$ and suppose that there is a $\delta > 1$ such that $|P^{(\alpha)}(\lambda)| \lesssim |\lambda|^{-\delta|\alpha|}|P(\lambda)|$ for $|\lambda|$ sufficiently large. Since P is of order $N > 0$ there exists a multi-index α with $|\alpha| = N$ such that $P^{(\alpha)}(\lambda) = c$ for some non-zero constant c. We have

$$|c| = |P^{(\alpha)}(\lambda)| \lesssim |\lambda|^{-\delta N}|P(\lambda)| \to 0$$

as $|\lambda| \to \infty$. This gives our contradiction, so there can be no polynomial of positive degree that satisfies the (C, δ)-estimate with $\delta > 1$. If $\delta = 1$ then the operator is elliptic. Indeed, the triangle inequality gives us

$$\frac{|\sigma_P(\lambda)|}{|\lambda|^N} \geq \left| \frac{|P(\lambda)|}{|\lambda|^N} - \frac{|\sigma_P(\lambda) - P(\lambda)|}{|\lambda|^N} \right|.$$

For $|\lambda|$ sufficiently large we know that there is some $C > 0$ such that $|P(\lambda)| \geq C|\lambda|^N$. Choosing $|\lambda|$ larger if needs be, we can make sure the second term on the right hand side is smaller than $C/2$. It follows that

$$|\sigma_P(\lambda)| \gtrsim |\lambda|^N$$

for $|\lambda|$ sufficiently large. But since $\sigma_P(\lambda)$ is homogeneous of degree N, it follows that $|\sigma_P(\lambda)| \gtrsim 1$ on S^{n-1} and so on all $\mathbf{R}^n$.

9. If $a \in \mathrm{Sym}(X, \mathbf{R}^k, N)$ show that $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbf{R}^k, N - |\beta|)$. If a_1, a_2 belong to $\mathrm{Sym}(X, \mathbf{R}^k; N_1)$ and $\mathrm{Sym}(X, \mathbf{R}^k; N_2)$ respectively, show that $a_1 a_2 \in \mathrm{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

Answer: If $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ then $D_x^\alpha D_\theta^\beta a$ is certainly smooth and for each compact $K \subset X$ we have

$$\left| D_x^{\alpha'} D_\theta^{\beta'} (D_x^\alpha D_\theta^\beta a) \right| = \left| D_x^{\alpha+\alpha'} D_\theta^{\beta+\beta'} a \right| \lesssim_{K,\alpha',\beta'} \langle \theta \rangle^{N - |\beta| - |\beta'|}$$

so $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbf{R}^k; N - |\beta|)$. For the next bit, for $a_i \in \mathrm{Sym}(X, \mathbf{R}^k; N_i)$ obviously $a_1 a_2$ is smooth. For the relevant estimate we use the multi-nomial Liebniz lemma

$$D_x^\alpha D_\theta^\beta (a_1 a_2) = \sum_{|\alpha'| \leq |\alpha|} \sum_{|\beta'| \leq |\beta|} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} D_x^{\alpha'} D_\theta^{\beta'} (a_1) D_x^{\alpha - \alpha'} D_\theta^{\beta - \beta'} (a_2).$$

Using the previous result we have for each compact $K \subset X$

$$\left| D_x^\alpha D_\theta^\beta (a_1 a_2) \right| \lesssim_{K,\alpha,\beta} \sum_{|\alpha'| \leq |\alpha|} \sum_{|\beta'| \leq |\beta|} \langle \theta \rangle^{N_1 - |\beta'|} \langle \theta \rangle^{N_2 - |\beta| + |\beta'|} \lesssim_{K,\alpha,\beta} \langle \theta \rangle^{N_1 + N_2 - |\beta|},$$

hence $a_1 a_2 \in \mathrm{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

10. Define the differential operator

$$L = \sum_{j=1}^k a_j(x, \theta) \frac{\partial}{\partial \theta_j} + \sum_{j=1}^n b_j(x, \theta) \frac{\partial}{\partial x_j} + c(x, \theta)$$

where $a_j \in \mathrm{Sym}(X; \mathbf{R}^k; 0)$, $b_j \in \mathrm{Sym}(X, \mathbf{R}^k; -1)$ and $c \in \mathrm{Sym}(X, \mathbf{R}^k; -1)$. Show that the formal adjoint L^* has the same form with coefficients in the same spaces of symbols.

Answer: The formal adjoint is given by

$$L^* = -\sum_{j=1}^k a_j(x, \theta) \frac{\partial}{\partial \theta_j} - \sum_{j=1}^n b_j(x, \theta) \frac{\partial}{\partial x_j} + \left(c(x, \theta) - \sum_{j=1}^k \frac{\partial a_j}{\partial \theta_j} - \sum_{j=1}^n \frac{\partial b_j}{\partial x_j} \right).$$

By the results in question **??** we see that this is the same form as L.

11. Consider the oscillatory integral defined by the phase function Φ and the symbol $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ where $X \subset \mathbf{R}^n$. Suppose that Φ is *non-degenerate*, i.e. the differentials $\mathrm{d}(\partial \Phi / \partial \theta_j)$, $(1 \leq j \leq k)$, are linearly independent. Prove that the set

$$M(\Phi) = \{(x, \theta) : x \in X, \theta \in \mathbf{R}^k \setminus \{0\}, \nabla_\theta \Phi(x, \theta) = 0\}$$

is an smooth n-dimensional submanifold of $X \times (\mathbf{R}^k \setminus \{0\})$. If you have experience with geometry, prove that the set

$$SP(\Phi) = \{(x, \nabla_x \varphi(x, \theta)) : (x, \theta) \in M(\Phi)\}$$

is a Lagrangian submanifold of $T^*X \setminus \{0\}$ (the cotangent bundle over X). If you don't have experience with geometry, but have access to the Internet, Google the phrase "Lagrangian submanifold".

Answer: The set $M(\Phi)$ is the subset of phase space $X \times \mathbf{R}^k$ such that

$$f_1(x,\theta) = 0, \quad f_2(x,\theta) = 0, \quad \dots \quad f_k(x,\theta) = 0,$$

where $f_i = \partial\Phi/\partial\theta_i$. Set $\mathbf{F} = (f_1, \dots, f_k)$. The condition in the question means that derivative

$$D\mathbf{F}(x,\theta) = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n & \partial f_1/\partial\theta_1 & \dots & \partial f_1/\partial\theta_k \\ \partial f_2/\partial x_1 & \cdots & \partial f_2/\partial x_n & \partial f_2/\partial\theta_1 & \dots & \partial f_2/\partial\theta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial f_k/\partial x_1 & \cdots & \partial f_k/\partial x_n & \partial f_k/\partial\theta_1 & \dots & \partial f_k/\partial\theta_k \end{pmatrix}$$

is of maximal rank, so the implicit function theorem tells us that $M(\Phi) = \{(x,\theta) : \mathbf{F} = 0\}$ defines a smooth (n-dimensional) submanifold of $X \times \mathbf{R}^k$. For the next bit, introduce local coordinates $(x,p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ for T^*X. Similar to before, we can show that $SP(\Phi)$ is a smooth n-dimensional submanifold of T^*X. To prove that it is Lagrangian we must prove that for each $\xi \in SP(\Phi)$ we have $\omega(X_\xi, Y_\xi) = 0$ for each $X_\xi, Y_\xi \in T_\xi SP(\Phi)$. Here ω is the usual symplectic form on T^*X given locally by

$$\omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}x_i = \mathrm{d}\left(\sum_{i=1}^n p_i \mathrm{d}x_i\right) \equiv \mathrm{d}\alpha$$

where α is the canonical 1-form. It is enough for us to show that ω vanishes on $SP(\Phi)$. For any $\xi \in SP(\Phi)$ we have $\xi = (x,p)$ with $p = \nabla_x\Phi$, so

$$\alpha\big|_{SP(\Phi)} = \sum_{i=1}^n \frac{\partial\Phi}{\partial x_i}\mathrm{d}x_i = \mathrm{d}\Phi - \sum_{i=1}^n \frac{\partial\Phi}{\partial\theta_i}\mathrm{d}\theta_i = \mathrm{d}\Phi$$

since $\nabla_\theta\Phi = 0$ on $SP(\Phi)$. But then $\omega = \mathrm{d}(\mathrm{d}\Phi) = 0$ on $SP(\Phi)$ since $\mathrm{d}^2 = 0$. We deduce that $SP(\Phi)$ is a Lagrangian submanifold of the cotangent bundle over X.

12. Let $p = (\omega, \mathbf{p})$ and $x = (t, \mathbf{x})$ be coordinates in $\mathbf{R}^4$. Show that the distribution[3] Δ_F defined by

$$\hat{\Delta}_F(p) = \lim_{\epsilon\to 0} \frac{1}{-\omega^2 + |\mathbf{p}|^2 + m^2 - \mathrm{i}\epsilon}$$

is a fundamental solution to the Klein-Gordon operator $\partial_t^2 - \Delta_{\mathbf{x}} + m^2$. Set $\omega_{\mathbf{p}} = \sqrt{m^2 + |\mathbf{p}|^2}$ and show

$$\Delta_F(x) = \frac{\mathrm{i}}{2(2\pi)^3} \int \frac{e^{-\mathrm{i}\omega_{\mathbf{p}}|t| + \mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{\omega_{\mathbf{p}}}\,\mathrm{d}\mathbf{p}$$

when interpreted as the sum of an ordinary function and an oscillatory integral. Show that the oscillatory integral has symbol belonging to $\mathrm{Sym}(\mathbf{R}^+ \times \mathbf{R}^3, \mathbf{R}^3; -1)$. Show that the singular support of the Feynman propagator is contained in the light cone $|t| = |\mathbf{x}|$.

Answer: This question is a bit too long – you've been warned. Using the Fourier transform on $\mathcal{S}'(\mathbf{R}^4)$ we know that

$$(\partial_t^2 - \Delta_{\mathbf{x}} + m^2)\Delta_F = \delta_0 \quad \Leftrightarrow \quad (-\omega^2 + |\mathbf{p}|^2 + m^2)\hat{\Delta}_F = 1,$$

so the first part is immediate. We have

$$\left\langle \hat{\Delta}_F, \varphi \right\rangle = \lim_{\epsilon\to 0} \iint \frac{\varphi(\omega,\mathbf{p})}{-\omega^2 + |\mathbf{p}|^2 + m^2 - \mathrm{i}\epsilon}\,\mathrm{d}\omega\,\mathrm{d}\mathbf{p}.$$

[3] This is the Feynman propagator. This distribution, and ones closely related to it, are of central importance in theoretical physics – see Quantum Field Theory courses.

We will interchange some integrals (valid by dominated convergence). Using $\tilde{\varphi}$ to denote the Fourier transform in the t-coordinates, the integral in the right hand side of the expression above is

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{\epsilon\to 0}\iint \frac{1}{-\omega^2+|\mathbf{p}|^2+m^2-\mathrm{i}\epsilon}\left(\int e^{\mathrm{i}\omega t}\tilde{\varphi}(t,\mathbf{p})\,\mathrm{d}t\right)\mathrm{d}\omega\,\mathrm{d}\mathbf{p}\\
&\quad=\frac{1}{2\pi}\lim_{\epsilon\to 0}\iint\left(\int \frac{e^{\mathrm{i}\omega t}}{-\omega^2+|\mathbf{p}|^2+m^2-\mathrm{i}\epsilon}\,\mathrm{d}\omega\right)\tilde{\varphi}(t,\mathbf{p})\,\mathrm{d}t\,\mathrm{d}\mathbf{p}\\
&\quad=\iint\left(\frac{\mathrm{i}e^{-\mathrm{i}\omega_{\mathbf{p}}|t|}}{2\omega_{\mathbf{p}}}\right)\tilde{\varphi}(t,\mathbf{p})\,\mathrm{d}t\,\mathrm{d}\mathbf{p}.
\end{aligned}$$

The final integral is done by closing contours in the upper or lower half complex ω-plane depending on whether t is positive or negative. In each case you pick up exactly one residue from the poles at

$$\omega=\pm(|\mathbf{p}|^2+m^2-\mathrm{i}\epsilon)^{1/2}.$$

After performing this calculation you can take $\epsilon\to 0$ using dominated convergence. Inserting an imaginary regularisation we deduce

$$\left\langle\hat{\Delta}_F,\varphi\right\rangle=\iint\left(\frac{\mathrm{i}}{2(2\pi)^3}\int\frac{e^{-\mathrm{i}\omega_{\mathbf{p}}|t|+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{\omega_{\mathbf{p}}}\,\mathrm{d}\mathbf{p}\right)\hat{\varphi}(t,\mathbf{x})\,\mathrm{d}t\,\mathrm{d}\mathbf{x},$$

where the inner-integral is defined in terms of an oscillatory integral. If we introduce $\chi\in\mathcal{D}(\mathbf{R}^3)$ with $\operatorname{supp}\chi$ contained in some small neighbourhood of the origin, we can write

$$\Delta_F(t,\mathbf{x})=\frac{\mathrm{i}}{2(2\pi)^3}\int\chi_1(\mathbf{p})\frac{e^{-\mathrm{i}\omega_{\mathbf{p}}|t|+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{\omega_{\mathbf{p}}}\,\mathrm{d}\mathbf{p}+\frac{\mathrm{i}}{2(2\pi)^3}\int(1-\chi_1(\mathbf{p}))\frac{e^{-\mathrm{i}\omega_{\mathbf{p}}|t|+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{\omega_{\mathbf{p}}}\,\mathrm{d}\mathbf{p}$$

where the first integral is just an ordinary function and the second is to be interpreted as an oscillatory integral with phase function and amplitude given by

$$\Phi(x,\mathbf{p})=\mathbf{p}\cdot\mathbf{x}-|\mathbf{p}||t|,\qquad a(x,\mathbf{p})=\frac{(1-\chi(\mathbf{p}))e^{\mathrm{i}(|\mathbf{p}|-\omega_{\mathbf{p}})|t|}}{\omega_{\mathbf{p}}}$$

($\mathbf{p}$ is playing the rôle of the usual θ coordinate in oscillatory integrals). Note that we couldn't make the obvious choice $\Phi(x,\mathbf{p})=\mathbf{p}\cdot\mathbf{x}-\omega_{\mathbf{p}}|t|$ because this function fails to be homogeneous in $\mathbf{p}$. We need to show that our choices really are phase and amplitude functions on $(X,\mathbf{R}^3)$ where $X=\mathbf{R}^+\times\mathbf{R}^3$. Firstly, it's clear that Φ is a phase function because it's smooth on $X\times(\mathbf{R}^3\setminus\{0\})$, it's homogeneous of degree one in $\mathbf{p}$ and $\mathrm{d}\Phi$ is non-vanishing on $X\times\mathbf{R}^3\setminus\{0\}$ since $\partial\Phi/\partial t=|\mathbf{p}|$. We are left to show $a(x,\mathbf{p})$ is a symbol of order -1. This is a bit of a mess – we need to compute the derivatives of $\mathbf{p}$ and show that for each compact $K\subset X$

$$|D_x^\alpha D_{\mathbf{p}}^\beta a(x,\mathbf{p})|\lesssim_{K,\alpha,\beta}\langle\mathbf{p}\rangle^{-1-|\beta|}.$$

You can do this by induction, but it's a long and arduous task (i.e. you wouldn't get an example this involved in the exam). I leave it for the keen. Finally, by our result in lectures we know that $\operatorname{sing\,supp}\Delta_F$ is contained in the subset of X where $\nabla_{\mathbf{p}}\Phi=0$ for some $\mathbf{p}\in\mathbf{R}^3\setminus\{0\}$. Hence

$$\operatorname{sing\,supp}\Delta_F\subset\left\{(t,\mathbf{x}):\mathbf{x}-\frac{\mathbf{p}}{|\mathbf{p}|}t=0\ \text{ for some }\mathbf{p}\neq 0\right\}=\{(t,\mathbf{x}):|\mathbf{x}|=|t|\},$$

which is the light cone. Note that this *does not* depend on m (which refers to the mass of a boson).