1. Write the following distributions as sums of derivatives of continuous functions

(a) \( \delta_0 \), (b) \( \text{sgn}(x) \), (c) p.f. \( \left( \frac{1}{x^2} \right) \).

**Answer:** Use your intuition and prove your guesses are correct. E.g. the first one is \((xH)''\). For the second use \(\text{sgn}(x) = H(x) - H(-x)\) with \(H = (xH)'\). The third is proportional \((\log |x|)''\).

2. For an \(N\)th order polynomial \(P\) in \(n\)-variables set \(P^{(\alpha)}(\lambda) = \partial^{\alpha}P(\lambda)\). Show that

\[
P(D)[\varphi \psi] = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} D^{\alpha}(\varphi)P^{(\alpha)}(D)(\psi)
\]

for all test functions \(\psi, \varphi \in \mathcal{D}(\mathbb{R}^n)\).

**Answer:** Note that it’s true for monomials (Leibniz rule), so it’s true for polynomials. Alternatively, using the Fourier transform it is sufficient to prove

\[
P(\lambda)(\hat{\varphi} \ast \hat{\psi})(\lambda) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} [(D^{\alpha}\varphi) \ast (P^{(\alpha)}(D)\psi)](\lambda)
\]

Written out in full, the right hand side is

\[
\int \left[ \sum_{|\alpha| \leq N} \frac{1}{\alpha!} (\lambda - \mu)^{\alpha} P^{(\alpha)}(\mu) \right] \hat{\varphi}(\lambda - \mu) \hat{\psi}(\mu) \, d\mu
\]

and by Taylor’s theorem the term in parenthesis is just \(P(\lambda)\) so we are done.

3. Let \(\{\Delta_i\}_{i \geq 1}\) and \(\{c_i\}_{i \geq 1}\) be as in the proof of the Malgrange-Ehrenpreis theorem. Prove that

\[
\varphi \mapsto \langle E, \varphi \rangle = \sum_{i=1}^{\infty} \int_{\Delta_i} \left( \int_{\text{Im}\, \lambda_n = c_i} \hat{\varphi}(-\lambda', -\lambda_n) \frac{d\lambda'}{P(\lambda', \lambda_n)} \right) d\lambda_n
\]

defines an element of \(\mathcal{D}'(\mathbb{R}^n)\).

**Answer:** Let \(\varphi \in \mathcal{D}(\mathbb{R}^n)\) with supp \(\varphi \subset K\). Since \(|P(\lambda', \lambda_n)| \gtrsim 1\) uniformly in each integrand we can have

\[
|\langle E, \varphi \rangle| \lesssim \sum_{i=1}^{\infty} \int_{\Delta_i} \int |\hat{\varphi}(-\lambda', -\lambda_n - ic_i)| \, d\lambda_n \, d\lambda'.
\]

Observe that

\[
|\lambda^\alpha \hat{\varphi}(-\lambda', -\lambda_n - ic_i)| = \left| \int e^{c_i \cdot x_n} \varphi(x) D_x^\alpha e^{i\lambda \cdot x} \, dx \right| = \left| \int e^{i\lambda \cdot x} D_x^\alpha [e^{c_i \cdot x_n} \varphi(x)] \, dx \right|
\]

Applying the derivatives and using a simple estimate on the integral we see that

\[
|\hat{\varphi}(-\lambda', -\lambda_n - ic_i)| \lesssim_{c_i,K} \langle \lambda \rangle^{-N} \sum_{|\alpha| \leq N} \sup_{|\alpha| \leq N} |\partial^{\alpha} \varphi| \lesssim_{K} \langle \lambda \rangle^{-N} \sum_{|\alpha| \leq N} \sup_{|\alpha| \leq N} |\partial^{\alpha} \varphi|,
\]

for any \(N \geq 0\). We were able to get rid of the \(c_i\) dependency because the \(c_i\) are the imaginary parts of complex numbers contained inside a fixed horizontal strip. Hence

\[
|\langle E, \varphi \rangle| \lesssim_{K} \left( \sum_{|\alpha| \leq N} \sup_{|\alpha| \leq N} |\partial^{\alpha} \varphi| \right) \sum_{i=1}^{\infty} \int_{\Delta_i} \int \langle \lambda \rangle^{-N} \, d\lambda_n \, d\lambda' \lesssim_{K} \sum_{|\alpha| \leq N} \sup_{|\alpha| \leq N} |\partial^{\alpha} \varphi|
\]

where \(N\) is chosen sufficiently large so the \(\lambda\)-integral converges. We deduce that \(E \in \mathcal{D}'(\mathbb{R}^n)\).
4. Let $A$ be a symmetric $n \times n$ matrix with $\det A \neq 0$ and $\text{Re} A \geq 0$ (i.e. $\text{Re} A$ is positive semi-definite). Construct, explicitly, a fundamental solution to the operator

$$L = \frac{\partial}{\partial t} - \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$ 

[Hint: if $A$ is the identity then $L$ is just the heat operator, so we know the fundamental solution in this case. The fundamental solution for the general case looks similar.]

**Answer:** Recall that $E(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ for $t > 0$ and zero elsewhere, is a fundamental solution when $A_{ij} = \delta_{ij}$, so we guess a fundamental solution for $t > 0$ of the form

$$E_A(x, t) = c_A t^{-n/2} \exp(-x \cdot Bx/4t)$$

and zero elsewhere, for some $B \in \text{Mat}_n(\mathbb{C})$ and constant $c_A$, both of which will obviously depend on $A$. For $E_A$ to be a fundamental solution for $L$ we certainly require $LE_A$ to vanish on $t > 0$. A straightforward computation shows that we require $BAB = B$, i.e. $B = A^{-1}$. We note that $\text{Re} B$ is positive semi-definite since $\text{Re} A$ is positive definite (exercise). We would like to show that

$$\langle LE_A, \varphi \rangle = \langle E_A, L^* \varphi \rangle = \int \int E_A L^* \varphi \, dx \, dt = \varphi(0)$$

where $L^* = -\partial_t - A_{ij} \partial_i \partial_j$ is the formal adjoint of $L$. Just as with the case with $A_{ij} = \delta_{ij}$ we did in lectures, we find that

$$\langle LE_A, \varphi \rangle = \lim_{\epsilon \to 0} \int E_A(x, \epsilon) \varphi(x, \epsilon) \, dx.$$ 

Usually we would make the substitution $x = e^{1/2}x'$ and take the limit, but the difficulty here lies in the fact that the integrand doesn’t converge absolutely as $\epsilon \downarrow 0$ if at least one of the $\sigma_i$ are purely imaginary, so we need to do more work if we want to take the limit inside the integral. If you don’t like fiddly limits, just assume $A$ is positive definite and take the limit as usual, else, read on. First we set

$$\varphi(Rx) = P(x)e^{-|x|^2} + \psi(x)$$

where $P$ is a polynomial of order $N$ (the $N$th Taylor polynomial of $e^{|x|^2}$) and $\psi$ vanishes to order $N+1$ at $x = 0$ (i.e. $\partial^\alpha \psi(0) = 0$ when $|\alpha| \leq N+1$). Note that $\psi$ goes to zero very quickly as $|x|$ gets large. Let us concentrate on the first term – we want to compute terms that look like

$$e^{-n/2} \int e^{-x \cdot Bx/4\epsilon} e^{-|x|^2} x^{2\alpha} \, dx$$

(we’ve discarded the parts of $P$ which contain an odd bit since these integrals vanish). Let us first do this integral for $\alpha = 0$. Using the fact that $\int e^{-x \cdot \Lambda x} \, dx = \pi^{n/2}(\det \Lambda)^{-1/2}$ for positive definite matrices. Letting $\sigma$ be a diagonal matrix with entries $\sigma_i > 0$ we know that

$$e^{-n/2} \int e^{-x \cdot Bx/4\epsilon} e^{-x \cdot \sigma x} \, dx = (4\pi)^{n/2} \det(B + \epsilon \sigma)$$

Assume first that $\Lambda$ is real and symmetric, so it can be diagonalised by an orthogonal transformation. Using this transformation the resulting integral can be written as the $n$–fold product of one dimensional integrals $\int e^{-\lambda_i x_i^2} \, dx$ where each $\lambda_i > 0$ corresponds to the $i$th eigenvalue of $\Lambda$. By Cauchy’s theorem this is equivalent to the limit as $R \to \infty$ of

$$2 \int_0^R e^{-\lambda_i x^2} \, dx = 2 \int_{\Gamma_R} e^{-\lambda_i z^2} \, dz + \frac{1}{2} \int_{C_R} e^{-\lambda_i z^2} \, dz$$

where $\Gamma_R$ is the ray in $\text{Re} z \geq 0$ with arg $z = -\frac{1}{2} \arg \lambda_i$ and $C_R$ is the arc which joins this ray back to the real axis. Since $\lambda_i x^2 = |\lambda_i||z|^2$ on $\Gamma_R$, this integral converges to $e^{-\arg \lambda_i /2} \sqrt{\pi / |\lambda_i|} \approx \pi^{1/2} \lambda_i^{-1/2}$ in the limit $R \to \infty$, where the square root is positive if $\lambda_i$ is real and positive. It is straightforward to show that the contribution from $C_R$ vanishes using a variation of Jordan’s lemma. We deduce that

$$\int e^{-x \cdot \Lambda x} \, dx = \pi^{n/2}(\lambda_1 \lambda_2 \cdots \lambda_n)^{-1/2} = \pi^{n/2}(\det \Lambda)^{-1/2}.$$ 

To extend this result for complex positive definite $\Lambda$, note that both sides of this result constitute analytic functions of $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ if $\Lambda$ is positive definite, so a simple analytic continuation argument extends the result.
from the previous result. We simply set $\sigma = I$ so it is relevant for our computation, so in the limit when $\alpha = 0$ we get $(4\pi)^{n/2}(\det B)^{-1/2}$. For $\alpha \neq 0$ we differentiate this result each side with respect to $\sigma_i$ as many times as we please and then set $\sigma = I$ at the end. We find

$$e^{-n/2} \int e^{-x \cdot Bz/4\epsilon} e^{-x \cdot \sigma x^2 \alpha} \, dx = \mathcal{O}(\epsilon^{n}).$$

So we have that

$$\langle LE_A, \varphi \rangle = \frac{(4\pi)^{n/2}c_A}{(\det B)^{1/2}} \varphi(0) + \lim_{\epsilon \to 0} \left( e^{-n/2} \int e^{-x \cdot Bz/4\epsilon} \psi(x) \, dx \right).$$

The remaining term is easy to deal with using an integration by parts argument. Set $L_i = -\frac{1}{2} A_{ij} \partial / \partial x_j$. Then assuming Einstein summation convention and using the symmetry of $B$ we find

$$L_i e^{-x \cdot Bz/4\epsilon} = (4\epsilon)^{-1} A_{ijk} x_k e^{-x \cdot Bz/4\epsilon} = (4\epsilon)^{-1} x_i e^{-x \cdot Bz/4\epsilon}.$$

This allows us to write

$$\int e^{-x \cdot Bz/4\epsilon} \psi(x) \, dx = 4\epsilon \int \frac{\psi(x)}{x_i} L_i e^{-x \cdot Bz/4\epsilon} \, dx \quad \text{(no sum)}.$$

We can keep applying this trick many times as we like since $\partial^\alpha \psi(0) = 0$ for $|\alpha| \leq N + 1$, where $N$ can be chosen as large as we need. Consequently, the remaining term vanishes in the limit. So we deduce that $E_A$ is indeed a fundamental solution for $L$ with constant $c_A = (4\pi)^{-n/2}(\det A)^{-1/2}$. Shazam!\(^2\)

5. Suppose $L = P(\partial / \partial x_1, \ldots, \partial / \partial x_n)$ is an $N$th order elliptic partial differential operator with constant coefficients. Given $(\lambda, c) \in S^{n-1} \times \mathbb{R}$, show that for $R$ sufficiently large the function

$$u(x; \lambda, c) = \frac{1}{2\pi i} \oint_{|z| = R} \frac{e^{z(x \cdot \lambda - c)}}{z P(z\lambda)} \, dz$$

satisfies $L[u] = 1$ with $u = 0$ on the surface $x \cdot \lambda = c$. Here the integral is taken in the complex $z$-plane around a circle centred at the origin with radius $R$. How big did $R$ need to be?

**Answer:** First choose $R > 0$ so that all the roots of the polynomial $p_\lambda(z) = zP(z\lambda)$ are contained inside $|z| < R$. Then the integral is well defined and we have

$$Lu(x) = \frac{1}{2\pi i} \oint_{|z| = R} \frac{P(z\lambda) e^{z(x \cdot \lambda - c)}}{z P(z\lambda)} \, dz = \frac{1}{2\pi i} \oint_{|z| = R} \frac{e^{z(x \cdot \lambda - c)}}{z} \, dz = 1$$

which follows from the residue theorem. We also have

$$u|_{x \cdot \lambda = c} = \frac{1}{2\pi i} \oint_{|z| = R} \frac{dz}{z P(z\lambda)} = 0$$

since we can push the contour of integration out to infinity.

6. Let $P$ be an $N$th order polynomial in $n$-variables and suppose that there exists $C, \delta > 0$ such that $|P^{(\alpha)}(\lambda)| \leq C|\lambda|^{-\delta|\alpha|}|P(\lambda)|$ for all multi-indices $\alpha$ when $|\lambda|$ is sufficiently large. Show

$$P(D)u \in H^s_{\text{loc}}(X) \Rightarrow u \in H^{s+\delta N}_{\text{loc}}(X).$$

Deduce that $P(D)$ is hypoelliptic, i.e. $P(D)u \in C^\infty(X) \Rightarrow u \in C^\infty(X)$.

\(^2\)This is a fairly tough problem, so don’t worry if you didn’t manage to get it out. It’s currently an open (and difficult) problem to show that a similar pointwise limit exists when the test function belongs to some Sobolev space $H^s(\mathbb{R}^n)$ and $A_{ij} = i\delta_{ij}$. If you’re interested in this innocent looking problem, Google the phrase “Pointwise convergence of initial data for Schrödinger operators”.

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Answer: We need to perturb our bootstrap argument from lectures. Fix \( \varphi \in \mathcal{D}(X) \) and choose \( \psi_0, \ldots, \psi_M \) with \( \text{supp} \psi_{i-1} \subset \text{supp} \psi_i \) and \( \psi_{i-1} = 1 \) on \( \text{supp} \psi_i \) for \( i = 1, \ldots, M \). We have \( \psi_0 u \in H^r(\mathbb{R}^n) \) for some \( t \in \mathbb{R} \).

\[
P(D)(\psi_1 u) = \psi_1 P(D)u + \sum_{\alpha \neq 0} D^\alpha \psi_1 P^{(\alpha)}(D)(\psi_0 u)
\]
since \( \psi_0 = 1 \) on \( \text{supp} \psi_1 \). We deduce that \( P(D)(\psi_1 u) \in H^{\tilde{A}_1}(\mathbb{R}^n) \) where \( \tilde{A}_1 = \min\{s, t - N + 1\} \). Hence

\[
\int |\lambda|^{2\tilde{A}_1} |P(\lambda)(\psi_1 u)(\lambda)|^2 \, d\lambda < \infty.
\]

Using our \((C, \delta)\)-estimate for \( P(\lambda) \) we deduce that

\[
\int |\lambda|^{2(\tilde{A}_1 + |\alpha|\delta)} |P^{(\alpha)}(\lambda)(\psi_1 u)(\lambda)|^2 \, d\lambda < \infty,
\]
i.e. \( P^{(\alpha)}(D)(\psi_1 u) \in H^{A_1(\alpha)}(\mathbb{R}^n) \) where \( A_1(\alpha) = \min\{s + |\alpha|\delta, t - N + 1 + |\alpha|\delta\} \). The next step is

\[
P(D)(\psi_2 u) = \psi_2 P(D)u + \sum_{\alpha \neq 0} D^\alpha \psi_2 P^{(\alpha)}(D)(\psi_1 u) \in H^s(\mathbb{R}^n) \oplus \left( \bigoplus_{0 < |\alpha| \leq N} H^{A_1(\alpha)}(\mathbb{R}^n) \right).
\]

We deduce \( P(D)(\psi_2 u) \in H^{\tilde{A}_2}(\mathbb{R}^n) \), where \( \tilde{A}_2 = \min\{s, s + \delta, t - N + 1 + \delta\} = \min\{s, t - N + 1 + \delta\} \). Again using the same argument as we did for \( \psi_1 u \) we find that \( P^{(\alpha)}(D)(\psi_2 u) \in H^{A_2(\alpha)}(\mathbb{R}^n) \) where

\[
A_2(\alpha) = \{s + \delta|\alpha|, t - N + 1 + (1 + |\alpha|\delta)\}.
\]

Continuing inductively we deduce that

\[
P^{(\alpha)}(D)(\psi_M u) \in H^{A_M(\alpha)}(\mathbb{R}^n), \quad \text{where } A_M(\alpha) = \min\{s + \delta|\alpha|, t - N + 1 + (M - 1 + |\alpha|\delta)\}.
\]

Now choose \( M \) sufficiently large so that \( s < t - N + 1 + (M - 1)\delta \). Then we find that

\[
P^{(\alpha)}(D)(\psi_M u) \in H^{s + \delta|\alpha|}(\mathbb{R}^n).
\]

Now choose \( \alpha \) with \( |\alpha| = N \) so that \( P^{(\alpha)}(\lambda) = \text{const} \neq 0 \). We deduce \( \psi_M u \in H^{s + \delta N}(\mathbb{R}^n) \), and since \( \psi_M = 1 \) on \( \text{supp} \varphi \), we deduce that \( \varphi u \) belongs to the same Sobolev space. Since \( \varphi \in \mathcal{D}(X) \) was arbitrary it follows that \( u \in H^{s + \delta N}_{\text{loc}}(X) \). Hypoellipticity now follows.

7. Give an example of an operator \( P(D) \) that is hypoelliptic but not elliptic.

Answer: The Heat operator \( \partial_t - \Delta \) is hypoelliptic, but not elliptic (look at its principle symbol). Take the case \( P = \partial_t - \partial_x^2 \) for simplicity. In this case the symbol is

\[
P(\lambda_1, \lambda_2) = i\lambda_1 + \lambda_2^2.
\]

So \( |P(\lambda)|^2 = \lambda_1^2 + \lambda_2^4 \) for \( \lambda \in \mathbb{R}^2 \). It’s now a straightforward exercise to show that

\[
\left| \frac{P^{(\alpha)}(\lambda)}{P(\lambda)} \right|^2 \leq C|\lambda|^{-|\alpha|},
\]

so that the previous theorem holds with \( \delta = 1/2 \).

8. Does there exist a polynomial \( P \) of positive degree that satisfies a \((C, \delta)\)-estimate given in question 6 with \( \delta > 1 \)? What can you say about \( P(D) \) if the polynomial \( P \) satisfies the \((C, \delta)\)-estimate with \( \delta = 1 \)?

Answer: Let \( P \) be an \( N \)th order \((N > 0)\) polynomial on \( \mathbb{R}^n \) and suppose that there is a \( \delta > 1 \) such that \( |P^{(\alpha)}(\lambda)| \lesssim |\lambda|^{-\delta|\alpha|}|P(\lambda)| \) for \( |\lambda| \) sufficiently large. Since \( P \) is of order \( N > 0 \) there exists a multi-index \( \alpha \) with \( |\alpha| = N \) such that \( P^{(\alpha)}(\lambda) = c \) for some non-zero constant \( c \). We have

\[
|c| = |P^{(\alpha)}(\lambda)| \lesssim |\lambda|^{-\delta N}|P(\lambda)| \to 0
\]
as $|\lambda| \to \infty$. This gives our contradiction, so there can be no polynomial of positive degree that satisfies the $(C, \delta)$-estimate with $\delta > 1$. If $\delta = 1$ then the operator is elliptic. Indeed, the triangle inequality gives us

$$\frac{|\sigma_P(\lambda)|}{|\lambda|^N} \geq \frac{|P(\lambda)|}{|\lambda|^N} - \frac{|\sigma_P(\lambda) - P(\lambda)|}{|\lambda|^N}.$$ 

For $|\lambda|$ sufficiently large we know that there is some $C > 0$ such that $|P(\lambda)| \geq C|\lambda|^N$. Choosing $|\lambda|$ larger if needs be, we can make sure the second term on the right hand side is smaller than $C/2$. It follows that

$$|\sigma_P(\lambda)| \gtrsim |\lambda|^N$$

for $|\lambda|$ sufficiently large. But since $\sigma_P(\lambda)$ is homogeneous of degree $N$, it follows that $|\sigma_P(\lambda)| \gtrsim 1$ on $S^{n-1}$ and so on all $\mathbb{R}^n$.

9. If $a \in \text{Sym}(X, \mathbb{R}^k; N)$ show that $D^n_a D^n_\theta a \in \text{Sym}(X, \mathbb{R}^k, N - |\beta|)$. If $a_1, a_2$ belong to $\text{Sym}(X, \mathbb{R}^k; N_1)$ and $\text{Sym}(X, \mathbb{R}^k; N_2)$ respectively, show that $a_1 a_2 \in \text{Sym}(X, \mathbb{R}^k; N_1 + N_2)$.

**Answer:** If $a \in \text{Sym}(X, \mathbb{R}^k; N)$ then $D^n_a D^n_\theta a$ is certainly smooth and for each compact $K \subset X$ we have

$$\left|D^n_a D^n_\theta (D^n_a D^n_\theta a) \right| \leq \left|D^n_{a^*} + \alpha \right| D^n_{\theta^*} + \beta' \left|\langle \theta \rangle^N - |\beta| - |\beta'| \right|$$

so $D^n_a D^n_\theta a \in \text{Sym}(X, \mathbb{R}^k; N - |\beta|)$. For the next bit, for $a_i \in \text{Sym}(X, \mathbb{R}^k; N_1)$ obviously $a_1 a_2$ is smooth. For the relevant estimate we use the multi-nomial Liebniz lemma

$$D^n_a D^n_\theta (a_1 a_2) = \sum_{|\alpha| \leq |\alpha'|} \sum_{|\beta| \leq |\beta'|} \left(\begin{array}{c} \alpha \\ \alpha' \end{array}\right) \left(\begin{array}{c} \beta \\ \beta' \end{array}\right) D^n_{a^*} D^n_{\theta^*} (a_1) D^n_{a^*} D^n_{\theta^*} (a_2).$$

Using the previous result we have for each compact $K \subset X$

$$\left|D^n_a D^n_\theta (a_1 a_2) \right| \lessapprox_{K, a, \beta} \sum_{|\alpha| \leq |\alpha'|} \sum_{|\beta| \leq |\beta'|} \left(\begin{array}{c} \alpha \\ \alpha' \end{array}\right) \left(\begin{array}{c} \beta \\ \beta' \end{array}\right) \left(\theta \right)^{N_1 - |\beta|} \left(\theta \right)^{N_2 - |\beta'|} \lessapprox_{K, a, \beta} \left(\theta \right)^{N_1 + N_2 - |\beta'|},$$

hence $a_1 a_2 \in \text{Sym}(X, \mathbb{R}^k; N_1 + N_2)$.

10. Define the differential operator

$$L = \sum_{j=1}^k a_j(x, \theta) \frac{\partial}{\partial \theta_j} + \sum_{j=1}^n b_j(x, \theta) \frac{\partial}{\partial x_j} + c(x, \theta)$$

where $a_j \in \text{Sym}(X; \mathbb{R}^k; 0)$, $b_j \in \text{Sym}(X; \mathbb{R}^k; -1)$ and $c \in \text{Sym}(X; \mathbb{R}^k; -1)$. Show that the formal adjoint $L^*$ has the same form with coefficients in the same spaces of symbols.

**Answer:** The formal adjoint is given by

$$L^* = -\sum_{j=1}^k a_j(x, \theta) \frac{\partial}{\partial \theta_j} - \sum_{j=1}^n b_j(x, \theta) \frac{\partial}{\partial x_j} + \left(\begin{array}{c} c(x, \theta) - \sum_{j=1}^k \frac{\partial a_j}{\partial \theta_j} - \sum_{j=1}^n \frac{\partial b_j}{\partial x_j} \end{array}\right).$$

By the results in question ?? we see that this is the same form as $L$.

11. Consider the oscillatory integral defined by the phase function $\Phi$ and the symbol $a \in \text{Sym}(X, \mathbb{R}^k; N)$ where $X \subset \mathbb{R}^n$. Suppose that $\Phi$ is non-degenerate, i.e. the differentials $d(\partial \Phi / \partial \theta_j)$, $(1 \leq j \leq k)$, are linearly independent. Prove that the set

$$M(\Phi) = \{(x, \theta) : x \in X, \theta \in \mathbb{R}^k \setminus \{0\}, \nabla \Phi(x, \theta) = 0\}$$

is an smooth $n$-dimensional submanifold of $X \times (\mathbb{R}^k \setminus \{0\})$. If you have experience with geometry, prove that the set

$$SP(\Phi) = \{(x, \nabla_x \Phi(x, \theta)) : (x, \theta) \in M(\Phi)\}$$

is a Lagrangian submanifold of $T^*X \setminus \{0\}$ (the cotangent bundle over $X$). If you don’t have experience with geometry, but have access to the Internet, Google the phrase “Lagrangian submanifold”.

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**Answer:** The set $M(\Phi)$ is the subset of phase space $X \times \mathbb{R}^k$ such that

$$f_1(x, \theta) = 0, \quad f_2(x, \theta) = 0, \quad \ldots \quad f_k(x, \theta) = 0,$$

where $f_i = \partial \Phi / \partial \theta_i$. Set $F = (f_1, \ldots, f_k)$. The condition in the question means that derivative

$$DF(x, \theta) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \theta_1} & \cdots & \frac{\partial f_1}{\partial \theta_k} \\
\frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \theta_1} & \cdots & \frac{\partial f_2}{\partial \theta_k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} & \frac{\partial f_k}{\partial \theta_1} & \cdots & \frac{\partial f_k}{\partial \theta_k}
\end{pmatrix}$$

is of maximal rank, so the implicit function theorem tells us that $M(\Phi) = \{(x, \theta) : F = 0\}$ defines a smooth ($n$-dimensional) submanifold of $X \times \mathbb{R}^k$. For the next bit, introduce local coordinates $(x, p) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ for $T^*X$. Similar to before, we can show that $SP(\Phi)$ is a smooth $n$-dimensional submanifold of $T^*X$. To prove that it is Lagrangian we must prove that for each $\xi \in SP(\Phi)$ we have $\omega(X_\xi, Y_\xi) = 0$ for each $X_\xi, Y_\xi \in T_\xi SP(\Phi)$. Here $\omega$ is the usual symplectic form on $T^*X$ given locally by

$$\omega = \sum_{i=1}^n dp_i \wedge dx_i = d \left( \sum_{i=1}^n p_i dx_i \right) \equiv \alpha$$

where $\alpha$ is the canonical 1-form. It is enough for us to show that $\omega$ vanishes on $SP(\Phi)$. For any $\xi \in SP(\Phi)$ we have $\xi = (x, p)$ with $p = \nabla_x \Phi$, so

$$\alpha|_{SP(\Phi)} = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i} dx_i = d \Phi - \sum_{i=1}^n \frac{\partial \Phi}{\partial \theta_i} d\theta_i = d \Phi$$

since $\nabla_\Phi = 0$ on $SP(\Phi)$. But then $\omega = d(\Phi) = 0$ on $SP(\Phi)$ since $d^2 = 0$. We deduce that $SP(\Phi)$ is a Lagrangian submanifold of the cotangent bundle over $X$.

12. Let $p = (\omega, p)$ and $x = (t, x)$ be coordinates in $\mathbb{R}^4$. Show that the distribution $\Delta_F$ defined by

$$\hat{\Delta}_F(p) = \lim_{\epsilon \to 0} \frac{1}{-\omega^2 + |p|^2 + m^2 - i\epsilon}$$

is a fundamental solution to the Klein-Gordon operator $\partial_t^2 - \Delta_x + m^2$. Set $\omega_p = \sqrt{m^2 + |p|^2}$ and show

$$\Delta_F(x) = \frac{i}{2(2\pi)^3} \int \frac{e^{-i\omega_p |t| + ip \cdot x}}{\omega_p} dp$$

when interpreted as the sum of an ordinary function and an oscillatory integral. Show that the oscillatory integral has symbol belonging to $\text{Sym}(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R}_3; -1)$. Show that the singular support of the Feynman propagator is contained in the light cone $|t| = |x|$.

**Answer:** This question is a bit too long – you’ve been warned. Using the Fourier transform on $\mathcal{S}'(\mathbb{R}^4)$ we know that

$$(\partial_t^2 - \Delta_x + m^2)\Delta_F = \delta_0 \iff (-\omega^2 + |p|^2 + m^2)\hat{\Delta}_F = 1,$$

so the first part is immediate. We have

$$\langle \hat{\Delta}_F, \varphi \rangle = \lim_{\epsilon \to 0} \int \int \frac{\varphi(\omega, p)}{-\omega^2 + |p|^2 + m^2 - i\epsilon} d\omega dp.$$
We will interchange some integrals (valid by dominated convergence). Using \( \tilde{\varphi} \) to denote the Fourier transform in the \( t \)-coordinates, the integral in the right hand side of the expression above is

\[
\frac{1}{2\pi} \lim_{\epsilon \to 0} \int \int -\omega^2 + |p|^2 + m^2 - i\epsilon \left( \int e^{i\omega t} \tilde{\varphi}(t, p) \, dt \right) \, d\omega \, dp
\]

\[
= \frac{1}{2\pi} \lim_{\epsilon \to 0} \int \left( \int -\omega^2 + |p|^2 + m^2 - i\epsilon \, d\omega \right) \varphi(t, p) \, dt \, dp
\]

\[
= \int \left( e^{-i\omega p|t|} \right) \frac{\varphi(t, p)}{2\omega_p} \, dt \, dp.
\]

The final integral is done by closing contours in the upper or lower half complex \( \omega \)-plane depending on whether \( t \) is positive or negative. In each case you pick up exactly one residue from the poles at

\[ \omega = \pm (|p|^2 + m^2 - i\epsilon)^{1/2}. \]

After performing this calculation you can take \( \epsilon \to 0 \) using dominated convergence. Inserting an imaginary regularisation we deduce

\[
\left\langle \hat{\Delta}_F, \varphi \right\rangle = \int \int \left( \frac{i}{2(2\pi)^3} \int \frac{e^{-i\omega_p|t| + ip \cdot x}}{\omega_p} \, dp \right) \varphi(t, x) \, dt \, dx,
\]

where the inner-integral is defined in terms of an oscillatory integral. If we introduce \( \chi \in \mathcal{D}(\mathbb{R}^3) \) with \( \text{supp} \chi \) contained in some small neighbourhood of the origin, we can write

\[
\Delta_F(t, x) = \frac{i}{2(2\pi)^3} \int \chi_1(p) \frac{e^{-i\omega_p|t| + ip \cdot x}}{\omega_p} \, dp + \frac{i}{2(2\pi)^3} \int (1 - \chi_1(p)) \frac{e^{-i\omega_p|t| + ip \cdot x}}{\omega_p} \, dp
\]

where the first integral is just an ordinary function and the second is to be interpreted as an oscillatory integral with phase function and amplitude given by

\[
\Phi(x, p) = p \cdot x - |p||t|, \quad a(x, p) = \frac{(1 - \chi(p))e^{i|p| - \omega_p|t|}}{\omega_p}
\]

(\( p \) is playing the role of the usual \( \theta \) coordinate in oscillatory integrals). Note that we couldn’t make the obvious choice \( \Phi(x, p) = p \cdot x - \omega_p|t| \) because this function fails to be homogeneous in \( p \). We need to show that our choices really are phase and amplitude functions on \( (X, \mathbb{R}^3) \) where \( X = \mathbb{R}^+ \times \mathbb{R}^3 \). Firstly, it’s clear that \( \Phi \) is a phase function because it’s smooth on \( X \times (\mathbb{R}^3 \setminus \{0\}) \), it’s homogeneous of degree one in \( p \) and \( d\Phi \) is non-vanishing on \( X \times \mathbb{R}^3 \setminus \{0\} \) since \( \partial \Phi / \partial t = |p| \). We are left to show \( a(x, p) \) is a symbol of order \(-1\). This is a bit of a mess – we need to compute the derivatives of \( p \) and show that for each compact \( K \subset X \)

\[
|D_p^\alpha D_x^\beta a(x, p)| \lesssim_{K, \alpha, \beta} \langle p \rangle^{-1-|\beta|}.
\]

You can do this by induction, but it’s a long and arduous task (i.e. you wouldn’t get an example this involved in the exam). I leave it for the keen. Finally, by our result in lectures we know that sing \( \text{supp} \Delta_F \) is contained in the subset of \( X \) where \( \nabla_p \Phi = 0 \) for some \( p \in \mathbb{R}^3 \setminus \{0\} \). Hence

\[
\text{sing supp} \Delta_F \subset \left\{ (t, x) : \frac{p}{|p|} t = 0 \text{ for some } p \neq 0 \right\} = \left\{ (t, x) : |x| = |t| \right\},
\]

which is the light cone. Note that this does not depend on \( m \) (which refers to the mass of a boson).