Mathematical Tripos Part III Lent 2017
Distributions, hand out 1: notation Dr A.C.L. Ashton

Elements of $\mathbf{R}^n$ will be denoted by $x, y...$ etc. so that

$$x = (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

We will use $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$ to denote the volume element on $\mathbf{R}^n$. The capital letters X, Y and Z will denote open subsets of $\mathbf{R}^n$, and K will denote a compact (closed and bounded) subset of $\mathbf{R}^n$. Integrals over all of $\mathbf{R}^n$ and over $X \subset \mathbf{R}^n$ will be denoted

$$\int [\cdot] \, \mathrm{d}x, \qquad \int_X [\cdot] \, \mathrm{d}x,$$

respectively. We will use multi-index notation in which the Greek letters $\alpha, \beta, \gamma, \ldots$ denote multi-indices, i.e.

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{Z}_n^+.$$

The multi-index notation is read as

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

For a multi-index α we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. In some instances we may write ∂_x^α to make clear what variable we are differentiating with respect to. We will use the operator $D = -\mathrm{i}\partial$ for reasons that will become apparent when we start looking at Fourier transforms. The *support* of a function will denoted by supp, so that

$$\mathrm{supp}(f) = \mathrm{cl}\left(\{x \in \mathbf{R}^n : f(x) \neq 0\}\right),$$

i.e. the closure of the set on which the function is non-zero. If f is absolutely integrable on X, i.e.

$$\int_X |f| \, \mathrm{d}x < \infty$$

we will write $f \in L^1(X)$. We will often want to pass limits through an integral sign and to do this we will refer to the *dominated convergence theorem*: if $\{f_m\}_{m \geq 1}$ are a sequence of integrable functions on X such that $f_m(x) \to f(x)$ pointwise and $|f_m(x)| \leq g(x)$ for each m for some $g \in L^1(X)$, then

$$\lim_{m \to \infty} \left[\int_X f_m \, \mathrm{d}x\right] = \int_X \left[\lim_{m \to \infty} f_m\right] \mathrm{d}x = \int_X f \, \mathrm{d}x.$$

So if you can "dominate" your sequence of functions by some integrable function, you can pass the limit under the integral sign. The proof is not important to us, but can be found any measure theory text.