Mathematical Tripos Part III Lent 2017
Distributions, hand out 2: distributions with point support Dr A.C.L. Ashton

Suppose you are told that $u \in \mathcal{D}'(\mathbf{R}^n)$ and $\operatorname{supp} u = \{0\}$ – what can we say about u? We will show

$$u = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha \delta_0$$

for some constants c_α. That is to say, if a distribution is supported at a single point, then it must be some finite combination of derivatives of Dirac measures supported at that point.

Since u is compactly supported, we know it has finite order, N say. Fix $\rho \in \mathcal{D}(\mathbf{R}^n)$ with $\rho = 0$ on $|x| > 1$ and $\rho = 1$ on a neighbourhood of $x = 0$.. Set $\rho_\epsilon(x) = \rho(x/\epsilon)$, which will be useful because $\operatorname{supp} \rho_\epsilon$ shrinks to $\{0\}$ as $\epsilon \downarrow 0$. For $\varphi \in \mathcal{D}(\mathbf{R}^n)$ Taylor's theorem gives

$$\varphi(x) = \sum_{|\alpha| \le N} \frac{x^\alpha}{\alpha!} \partial^\alpha \varphi(0) + \psi(x)$$

where $\partial^\alpha \psi(0) = 0$ for $|\alpha| \le N$. Using this Taylor expansion we get

$$\langle u, \varphi \rangle = \sum_{|\alpha| \le N} \frac{\partial^\alpha \varphi(0)}{\alpha!} \langle u, x^\alpha \rangle + \langle u, \rho_\epsilon \psi \rangle ,$$

where we inserted the ρ_ϵ in the latter term, which is allowed since $\rho_\epsilon = 1$ on $\operatorname{supp} u$ for each $\epsilon > 0$. Let us estimate the final term. Since $\operatorname{supp} u = \{0\}$ and $\operatorname{ord}(u) = N$, there is some compact K containing $x = 0$ such that

$$| \langle u, \rho_\epsilon \psi \rangle | \lesssim \sum_{|\alpha| \le N} \sup_K |\partial^\alpha (\rho_\epsilon \psi)| \lesssim \sum_{|\alpha|+|\beta| \le N} \sup |\partial^\alpha \rho_\epsilon| |\partial^\beta \psi|.$$

Notice that $|\partial^\alpha \rho_\epsilon| \lesssim \epsilon^{-|\alpha|}$ and Taylor's theorem gives $|\partial^\beta \psi| \lesssim |x|^{N+1-|\beta|}$ since, as we noted earlier, $\partial^\beta \psi(0) = 0$ if $|\beta| \le N$. This latter estimate can be seen by looking at the remainder term in Taylor's theorem given in example sheet 1, question 2. In particular, on $\operatorname{supp} \rho_\epsilon$ we have $|\partial^\beta \psi| \lesssim \epsilon^{N+1-|\beta|}$, and consequently

$$| \langle u, \rho_\epsilon \psi \rangle | \lesssim \sum_{|\alpha|+|\beta| \le N} \epsilon^{-|\alpha|} \epsilon^{N+1-|\beta|} \lesssim \epsilon.$$

Taking the limit as $\epsilon \downarrow 0$ we conclude that

$$\langle u, \varphi \rangle = \sum_{|\alpha| \le N} \frac{\partial^\alpha \varphi(0)}{\alpha!} \langle u, \rho x^\alpha \rangle = \sum_{|\alpha| \le N} c_\alpha \langle \partial^\alpha \delta_0, \varphi \rangle$$

where $c_\alpha = (-1)^{|\alpha|} \langle u, x^\alpha \rangle / \alpha!$ which is just some set of constants (that obviously depend on u). So u is a finite combination of derivatives of Dirac measures, as we thought.