Mathematical Tripos Part IA Lent 2020
Vector Calculus, hand out 1: coordinate systems Dr A.C.L. Ashton

Cylindrical polar coordinates

The orthogonal curvilinear coordinates (ρ, ϕ, z) are related to standard Cartesian axes by

$$\mathbf{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}, \quad 0 \le \rho < \infty, \quad 0 \le \phi < 2\pi, \quad -\infty < z < \infty.$$

By differentiating with respect to (ρ, ϕ, z) and normalising we get the orthonormal basis vectors

$$\mathbf{e}_\rho = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and the corresponding scale factors are $\{h_\rho, h_\phi, h_z\} = \{1, \rho, 1\}$. So in this coordinate system we have $\mathbf{x} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$, while the line element and gradient operator become

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho\,\mathbf{e}_\rho + \rho\,\mathrm{d}\phi\,\mathbf{e}_\phi + \mathrm{d}z\,\mathbf{e}_z, \qquad \nabla = \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z}.$$

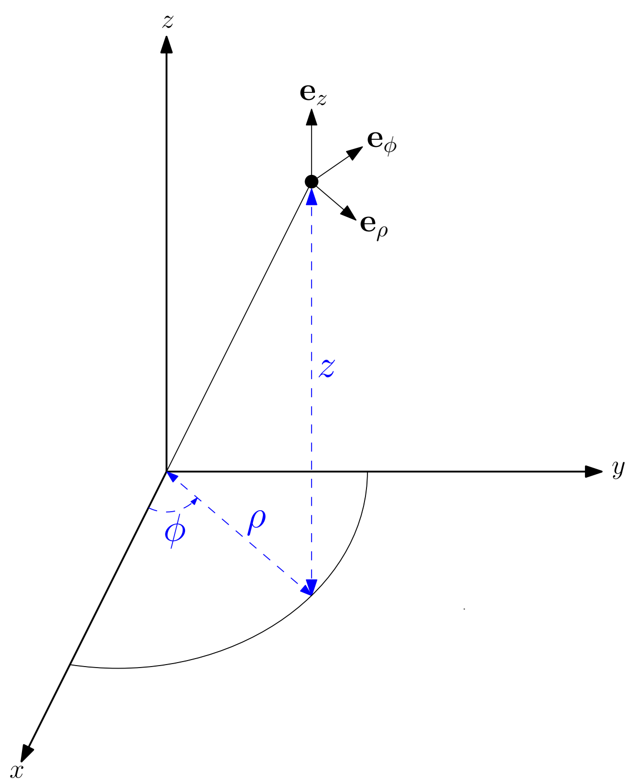

Diagram depicting cylindrical polars coordinates. Notice that $\mathbf{e}_\rho$ points directly outwards from the z-axis, whereas $\mathbf{e}_\phi$ traces out horizontal circles around the z-axis.

Spherical polar coordinates

The orthogonal curvilinear coordinates (r, θ, ϕ) are related to standard Cartesian axes by

$$\mathbf{x}(r,\theta,\phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}, \quad 0 \le r < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi.$$

By differentiating with respect to (r, θ, ϕ) and normalising we get the orthonormal basis vectors

$$\mathbf{e}_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix},$$

and the corresponding scale factors are $\{h_r, h_\theta, h_\phi\} = \{1, r, r\sin\theta\}$. So in this coordinate system we have $\mathbf{x} = r\mathbf{e}_r$, while the line element and gradient operator become

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi, \qquad \nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

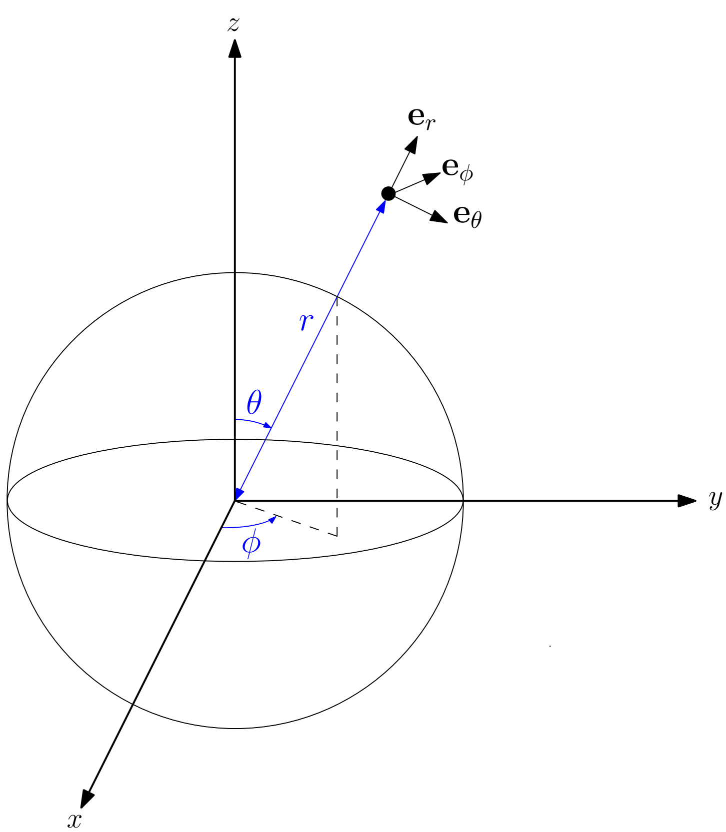

Diagram depicting spherical polars coordinates. Notice that $\mathbf{e}_r$ points directly outwards from the origin, $\mathbf{e}_\theta$ traces out a circle of constant longitude and $\mathbf{e}_\phi$ traces out a circle of constant latitude.