Mathematical Tripos Part IA Lent 2020
Vector Calculus, hand out 2: change of variables in 2D integrals Dr A.C.L. Ashton

Statement of result:
Let $x = x(u, v)$, $y = y(u, v)$ be a smooth, invertible transformation with smooth inverse that maps the region D' in the (u, v)-plane in a one-to-one fashion to the region D in the (x, y)-plane. Then

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v,$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \equiv J,$$

is called the Jacobian determinant. Informally we can write this result as $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$.

Definition of integral:
Recall our definition of the integral over a planar region D. We partition D with disjoint sets A_{ij}, of area δA_{ij}, so that each one is contained in a disc of radius $\epsilon > 0$. Choosing a point (x_i, y_j) in each A_{ij}, we define

$$\int_D f\,\mathrm{d}A = \lim_{\epsilon\to 0} \sum_{i,j} f(x_i, y_j)\,\delta A_{ij}. \qquad (\star)$$

We say the integral exists if the limit exists and is independent of the partition $\{A_{ij}\}$ and the points (x_i, y_j) chosen in each cell.

A rectangular partition:
In lectures we have shown that one way of computing the integral is to choose $\{A_{ij}\}$ as simple rectangular blocks of area $\delta A_{ij} = \delta x\,\delta y$, then split D into either vertical or horizontal slices. This resulted in

$$\int_D f\,\mathrm{d}A = \int_X \left(\int_{Y_x} f(x, y)\,\mathrm{d}y\right) \mathrm{d}x = \int_Y \left(\int_{X_y} f(x, y)\,\mathrm{d}x\right) \mathrm{d}y,$$

as seen in Figure 1. Here $X_y = \{x : (x, y) \in D\}$ and $Y_x = \{y : (x, y) \in D\}$. For short we just write

$$\int_D f\,\mathrm{d}A = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y. \qquad (\dagger)$$

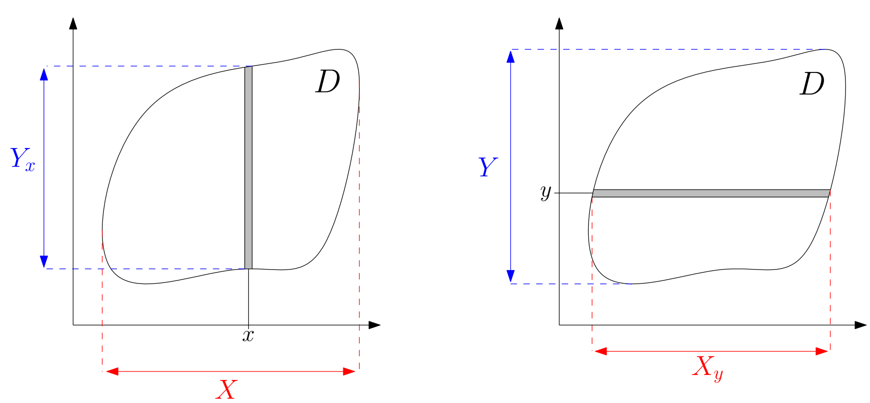


Figure 1: Performing $\int_D f\,\mathrm{d}A$ using vertical slices (left) and horizontal slices (right).

A non-rectangular partition:
Now rather than using a simple partition of D consisting of rectangles, we can instead choose a partition that is generated from the lines in the (x, y)-plane of the form[1]

$$u(x, y) = u_i, \quad v(x, y) = v_j$$

where the (u_i, v_j) are grid points in D' with $u_{i+1} - u_i = \delta u$ and $v_{j+1} - v_j = \delta v$ and $\max\{\delta u, \delta v\} \ll \epsilon$. So our partition is the *image* of a rectangular partition of D'. This is depicted in Figure **??**.

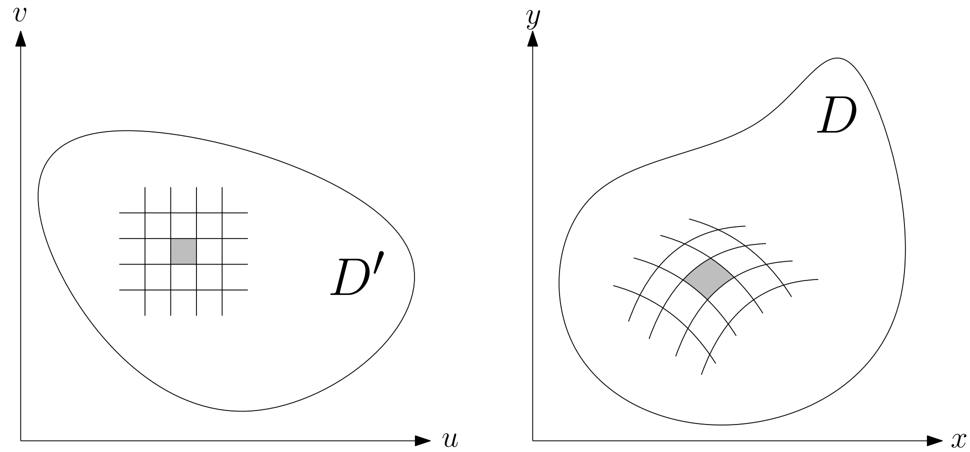


Figure 2: Lines of constant u and v in the (u, v)-plane transformed into the (x, y)-plane.

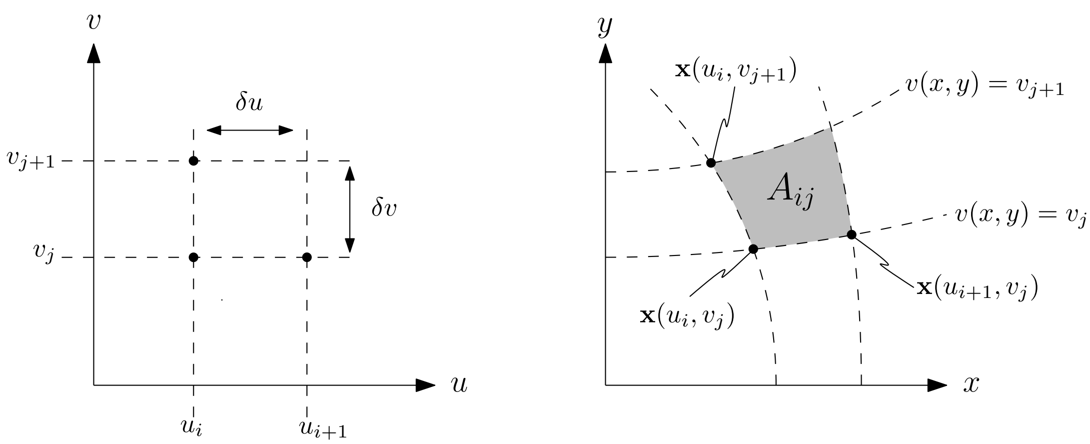


Figure 3: The set A_{ij} which forms part of our partition of D.

To compute the right hand side of $(\star)$ with respect to this partition, we choose sample points $(x_i, y_j) = (x(u_i, v_j), y(u_i, v_j))$ in each A_{ij}. The (i, j)-th entry in the sum $(\star)$ then becomes

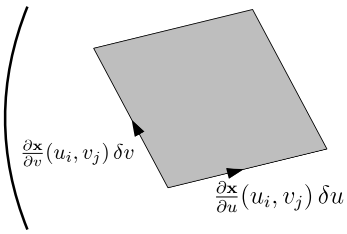
$$f(x(u_i, v_j), y(u_i, v_j))\, \delta A_{ij} \equiv f(\mathbf{x}(u_i, v_j))\, \delta A_{ij}.$$

It remains to estimate the area δA_{ij}. We note that, to leading order, the area δA_{ij} can be approximated by the parallelogram generated by the vectors

$$\mathbf{x}(u_{i+1}, v_j) - \mathbf{x}(u_i, v_j) \approx \frac{\partial \mathbf{x}}{\partial u}(u_i, v_j)\delta u, \quad \mathbf{x}(u_i, v_{j+1}) - \mathbf{x}(u_i, v_j) \approx \frac{\partial \mathbf{x}}{\partial v}(u_i, v_j)\delta v.$$

[1] We've assumed the map $x = x(u, v)$, $y = y(u, v)$ is invertible, so we can invert the relationship to get (u, v) as functions of (x, y), i.e. $u = u(x, y)$ and $v = v(x, y)$.

since $u_{i+1} = u_i + \delta u$ and $v_{j+1} = v_j + \delta v$. Recalling that the area of a parallelogram generated by the vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $|\det(\mathbf{a}\,|\,\mathbf{b})|$, we get the estimate



$$\delta A_{ij} \approx \text{area}\left(\quad \right) = \left|\det\left(\frac{\partial \mathbf{x}}{\partial u}(u_i, v_j) \,\middle|\, \frac{\partial \mathbf{x}}{\partial v}(u_i, v_j)\right)\right| \delta u\, \delta v$$

We recognise the term on the right hand side as being precisely $|J(u_i, v_i)|\,\delta u\,\delta v$. So with respect to this partition $\{A_{ij}\}$ of D we find

$$\begin{aligned}
\int_D f\,\mathrm{d}A &= \lim_{\epsilon\to 0} \sum_{i,j} f(x_i, y_j)\,\delta A_{ij} \\
&= \lim_{\epsilon\to 0} \sum_{i,j} f(x(u_i, v_j), y(u_i, v_j))|J(u_i, v_j)|\,\delta u\,\delta v \\
&= \iint_{D'} f(x(u,v), y(u,v))|J(u,v)|\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Since we know the integral, if it exists, must be independent of the partition chosen, we can compare this result to (†) and conclude

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(x(u,v), y(u,v))|J(u,v)|\,\mathrm{d}u\,\mathrm{d}v.$$

A note for intuition:
Using differentials, which measure first order changes, we have

$$\mathrm{d}x = \frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v, \quad \mathrm{d}y = \frac{\partial y}{\partial u}\,\mathrm{d}u + \frac{\partial y}{\partial v}\,\mathrm{d}v.$$

Or, if we use the vector notation $\mathrm{d}\mathbf{x} = \left(\begin{smallmatrix}\mathrm{d}x \\ \mathrm{d}y\end{smallmatrix}\right)$ and $\mathrm{d}\mathbf{u} = \left(\begin{smallmatrix}\mathrm{d}u \\ \mathrm{d}v\end{smallmatrix}\right)$ we get

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \mathrm{d}\mathbf{u}. \tag{‡}$$

Recall from IA Vectors and Matrices that the modulus of the determinant of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ tells you how areas scale under the transformation. Applying a linear transformation T to a box of area $\delta A'$ will produce a parallelogram of area $\delta A = |\det(T)|\delta A'$. In the (u, v)-plane we know $\mathrm{d}A' = \mathrm{d}u\,\mathrm{d}v$ and in the (x, y)-plane we have $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, so from the discussion above and the relation (‡) we expect

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}A' = |J|\,\mathrm{d}u\,\mathrm{d}v.$$

The argument we've given in the previous part of this handout is to convince the more sceptical amongst you.