Statement of result:
Let \( x = x(u, v) \), \( y = y(u, v) \) be a smooth, invertible transformation with smooth inverse that maps the region \( D' \) in the \((u, v)\)-plane in a one-to-one fashion to the region \( D \) in the \((x, y)\)-plane. Then

\[
\int\int_D f(x, y) \, dx \, dy = \int\int_{D'} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv,
\]

where

\[
\frac{\partial(x, y)}{\partial(u, v)} = \text{det} \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \equiv J,
\]

is called the Jacobian determinant. Informally we can write this result as \( dx \, dy = |J| \, du \, dv \).

Definition of integral:
Recall our definition of the integral over a planar region \( D \). We partition \( D \) with disjoint sets \( A_{ij} \), of area \( \delta A_{ij} \), so that each one is contained in a disc of radius \( \epsilon > 0 \). Choosing a point \((x_i, y_j)\) in each \( A_{ij} \), we define

\[
\int_D f \, dA = \lim_{\epsilon \to 0} \sum_{i,j} f(x_i, y_j) \, \delta A_{ij}. \tag{*}
\]

We say the integral exists if the limit exists and is independent of the partition \( \{A_{ij}\} \) and the points \((x_i, y_j)\) chosen in each cell.

A rectangular partition:
In lectures we have shown that one way of computing the integral is to choose \( \{A_{ij}\} \) as simple rectangular blocks of area \( \delta A_{ij} = \delta x \, \delta y \), then split \( D \) into either vertical or horizontal slices. This resulted in

\[
\int_D f \, dA = \int_X \left( \int_{Y_x} f(x, y) \, dy \right) \, dx = \int_Y \left( \int_{X_y} f(x, y) \, dx \right) \, dy,
\]

as seen in Figure 1. Here \( X_y = \{x : (x, y) \in D\} \) and \( Y_x = \{y : (x, y) \in D\} \). For short we just write

\[
\int_D f \, dA = \int\int_D f(x, y) \, dx \, dy. \tag{†}
\]

Figure 1: Performing \( \int_D f \, dA \) using vertical slices (left) and horizontal slices (right).
A non-rectangular partition:

Now rather than using a simple partition of $D$ consisting of rectangles, we can instead choose a partition that is generated from the lines in the $(x, y)$-plane of the form

$$u(x, y) = u_i, \quad v(x, y) = v_j$$

where the $(u_i, v_j)$ are grid points in $D'$ with $u_{i+1} - u_i = \delta u$ and $v_{j+1} - v_j = \delta v$ and $\max\{\delta u, \delta v\} \ll \epsilon$.

So our partition is the image of a rectangular partition of $D'$. This is depicted in Figure 2.

![Figure 2: Lines of constant $u$ and $v$ in the $(u, v)$-plane transformed into the $(x, y)$-plane.](image)

To compute the right hand side of $(\star)$ with respect to this partition, we choose sample points $(x_i, y_j) = (x(u_i, v_j), y(u_i, v_j))$ in each $A_{ij}$. The $(i, j)$-th entry in the sum $(\star)$ then becomes

$$f(x(u_i, v_j), y(u_i, v_j)) \delta A_{ij} = f(x(u_i, v_j)) \delta A_{ij}.$$ 

It remains to estimate the area $\delta A_{ij}$. We note that, to leading order, the area $\delta A_{ij}$ can be approximated by the parallelogram generated by the vectors

$$x(u_{i+1}, v_j) - x(u_i, v_j) \approx \frac{\partial x}{\partial u}(u_i, v_j)\delta u, \quad x(u_i, v_{j+1}) - x(u_i, v_j) \approx \frac{\partial x}{\partial v}(u_i, v_j)\delta v.$$ 

1We’ve assumed the map $x = x(u, v), y = y(u, v)$ is invertible, so we can invert the relationship to get $(u, v)$ as functions of $(x, y)$, i.e. $u = u(x, y)$ and $v = v(x, y)$. 

![Figure 3: The set $A_{ij}$ which forms part of our partition of $D$.](image)
since \( u_{i+1} = u_i + \delta u \) and \( v_{j+1} = v_j + \delta v \). Recalling that the area of a parallelogram generated by the vectors \( \mathbf{a} \) and \( \mathbf{b} \) is given by \( |\text{det}(\mathbf{a} | \mathbf{b})| \), we get the estimate

\[
\delta A_{ij} \approx \text{area} \left( \begin{array}{cc}
\frac{\partial x}{\partial u}(u_i, v_j) \\
\frac{\partial x}{\partial v}(u_i, v_j)
\end{array} \right) \delta u \delta v
\]

We recognise the term on the right hand side as being precisely \( |J(u_i, v_i)| \delta u \delta v \). So with respect to this partition \( \{A_{ij}\} \) of \( D \) we find

\[
\int_D f \, dA = \lim_{\epsilon \to 0} \sum_{i,j} f(x_i, y_j) \delta A_{ij}
\]

\[
= \lim_{\epsilon \to 0} \sum_{i,j} f(x(u_i, v_j), y(u_i, v_j)) |J(u_i, v_j)| \delta u \delta v
\]

\[
= \iint_{D'} f(x(u, v), y(u, v)) |J(u, v)| \, du \, dv.
\]

Since we know the integral, if it exists, must be independent of the partition chosen, we can compare this result to (†) and conclude

\[
\iint_D f(x, y) \, dx \, dy = \iint_{D'} f(x(u, v), y(u, v)) |J(u, v)| \, du \, dv.
\]

**A note for intuition:**

Using differentials, which measure first order changes, we have

\[
dx = \frac{\partial x}{\partial u} \, du + \frac{\partial x}{\partial v} \, dv, \quad dy = \frac{\partial y}{\partial u} \, du + \frac{\partial y}{\partial v} \, dv.
\]

Or, if we use the vector notation \( d\mathbf{x} = \left( \frac{dx}{dy} \right) \) and \( d\mathbf{u} = \left( \frac{du}{dv} \right) \) we get

\[
d\mathbf{x} = \left( \frac{\partial x}{\partial u} \ \frac{\partial x}{\partial v} \right) d\mathbf{u}. \quad (\dagger)
\]

Recall from IA Vectors and Matrices that the modulus of the determinant of a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) tells you how areas scale under the transformation. Applying a linear transformation \( T \) to a box of area \( \delta A' \) will produce a parallelogram of area \( \delta A = |\text{det}(T)| \delta A' \). In the \((u, v)\)-plane we know \( dA' = du \, dv \) and in the \((x, y)\)-plane we have \( dA = dx \, dy \), so from the discussion above and the relation \((\dagger)\) we expect

\[
dA = dx \, dy = |J| \, dA' = |J| \, du \, dv.
\]

The argument we’ve given in the previous part of this handout is to convince the more sceptical amongst you.