Mathematical Tripos Part IA Lent 2020
Vector Calculus, hand out 3: divergence and curl formulae Dr A.C.L. Ashton

Definitions:
For a vector field $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ we define the *divergence* of $\mathbf{F}$ by

$$\operatorname{div} \mathbf{F} \coloneqq \nabla \cdot \mathbf{F}.$$

We define the *curl* of $\mathbf{F}$ by

$$\operatorname{curl} \mathbf{F} \coloneqq \nabla \times \mathbf{F}.$$

For a scalar function $f : \mathbf{R}^3 \to \mathbf{R}$ we define the *Laplacian* of f by

$$\nabla^2 f \coloneqq \nabla \cdot \nabla f.$$

Cartesian coordinates:
For the vector field $\mathbf{F} = F_i \mathbf{e}_i$ we have

$$\nabla \cdot \mathbf{F} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot (F_j \mathbf{e}_j) = (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial F_j}{\partial x_i},$$

and using $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ we find that in Cartesian coordinates

$$\boxed{\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}}$$

Note that in our computation the derivatives just pass through the $\{\mathbf{e}_j\}$ because these basis vectors are independent of (x, y, z). Of course, this is not the case with a generic curvilinear coordinate system! The divergence of a vector field is a scalar function.

The curl is

$$\nabla \times \mathbf{F} = \left(\mathbf{e}_j \frac{\partial}{\partial x_j} \right) \times (F_k \mathbf{e}_k) = (\mathbf{e}_j \times \mathbf{e}_k) \frac{\partial F_k}{\partial x_j} = \mathbf{e}_i \left(\epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \right),$$

where we used $\mathbf{e}_j \times \mathbf{e}_k = \epsilon_{ijk} \mathbf{e}_i$. So in Cartesian coordinates

$$\boxed{[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}}$$

We can also express this as a formal determinant

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

Note that the definition of the divergence holds in any number of dimensions, but the definition of curl is special to three dimensions because it involves the vector product.

Using $F_i = \partial f / \partial x_i$ in the expression for the divergence $\nabla \cdot \mathbf{F}$ we get the following expression for the Laplacian in Cartesian coordinates

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}}$$

Curvilinear coordinates:
Let (u, v, w) be general orthogonal curvilinear coordinates. Then, in principle, we could compute

$$\nabla \cdot \mathbf{F} = \left(\mathbf{e}_u \frac{1}{h_u} \frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v} \frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w} \frac{\partial}{\partial w} \right) \cdot \left(F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w \right),$$

and similarly for $\nabla \times \mathbf{F}$ and $\nabla^2 f$. However, this turns out to be incredibly messy because, in the general case, the vectors $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ will *depend on* the coordinates (u, v, w), so we can't pass derivatives through them as we did with Cartesian coordinates. We simply state the results:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_w h_u F_v) + \frac{\partial}{\partial w}(h_u h_v F_w) \right],$$

$$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix},$$

$$\nabla^2 f = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_w h_u}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial f}{\partial w} \right) \right].$$

If all the scale factors are equal to 1 we recover the Cartesian version of these formulas.

Cylindrical polar coordinates:
In cylindrical polar coordinates (ρ, ϕ, z) with $(h_\rho, h_\phi, h_z) = (1, \rho, 1)$ we get

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z},$$

$$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix},$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Spherical polar coordinates:
In cylindrical polar coordinates (r, θ, ϕ) with $(h_r, h_\theta, h_\phi) = (1, r, r \sin\theta)$ we get

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \, F_\theta) + \frac{1}{r \sin\theta} \frac{\partial F_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & r \sin\theta \, \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta \, F_\phi \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}.$$