Definitions:
For a vector field $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ we define the divergence of $\mathbf{F}$ by

$$\text{div} \mathbf{F} := \nabla \cdot \mathbf{F}.$$ 

We define the curl of $\mathbf{F}$ by

$$\text{curl} \mathbf{F} := \nabla \times \mathbf{F}.$$ 

For a scalar function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ we define the Laplacian of $f$ by

$$\nabla^2 f := \nabla \cdot \nabla f.$$ 

Cartesian coordinates:
For the vector field $\mathbf{F} = F_i \mathbf{e}_i$ we have

$$\nabla \cdot \mathbf{F} = \left( \mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot (F_j \mathbf{e}_j) = (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial F_j}{\partial x_i},$$

and using $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ we find that in Cartesian coordinates

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}.$$ 

Note that in our computation the derivatives just pass through the $\{\mathbf{e}_j\}$ because these basis vectors are independent of $(x, y, z)$. Of course, this is not the case with a generic curvilinear coordinate system! The divergence of a vector field is a scalar function.

The curl is

$$\nabla \times \mathbf{F} = \left( \mathbf{e}_j \frac{\partial}{\partial x_j} \right) \times (F_k \mathbf{e}_k) = (\mathbf{e}_j \times \mathbf{e}_k) \frac{\partial F_k}{\partial x_j} = \mathbf{e}_i \left( \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \right),$$

where we used $\mathbf{e}_j \times \mathbf{e}_k = \epsilon_{ijk} \mathbf{e}_i$. So in Cartesian coordinates

$$[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}.$$ 

We can also express this as a formal determinant

$$\nabla \times \mathbf{F} = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$ 

Note that the definition of the divergence holds in any number of dimensions, but the definition of curl is special to three dimensions because it involves the vector product.

Using $F_i = \partial f / \partial x_i$ in the expression for the divergence $\nabla \cdot \mathbf{F}$ we get the following expression for the Laplacian in Cartesian coordinates

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}.$$
Curvilinear coordinates:
Let \((u, v, w)\) be general orthogonal curvilinear coordinates. Then, in principle, we could compute
\[
\nabla \cdot \mathbf{F} = \left( \frac{1}{h_u} \frac{\partial}{\partial u} \right) \left( h_u h_w F_u \right) + \left( \frac{1}{h_v} \frac{\partial}{\partial v} \right) \left( h_w h_u F_v \right) + \left( \frac{1}{h_w} \frac{\partial}{\partial w} \right) \left( h_u h_v F_w \right),
\]
and similarly for \(\nabla \times \mathbf{F}\) and \(\nabla^2 f\). However, this turns out to be incredibly messy because, in the general case, the vectors \(\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}\) will depend on the coordinates \((u, v, w)\), so we can’t pass derivatives through them as we did with Cartesian coordinates. We simply state the results:
\[
\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w} \left[ \frac{\partial}{\partial u} \left( h_v h_w F_u \right) + \frac{\partial}{\partial v} \left( h_w h_u F_v \right) + \frac{\partial}{\partial w} \left( h_u h_v F_w \right) \right],
\]
\[
\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w} \left| \begin{array}{ccc}
h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\
\frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\
h_u F_u & h_v F_v & h_w F_w
\end{array} \right|,
\]
\[
\nabla^2 f = \frac{1}{h_u h_v h_w} \left[ \frac{\partial}{\partial u} \left( h_v h_w \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left( h_w h_u \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left( h_u h_v \frac{\partial f}{\partial w} \right) \right].
\]
If all the scale factors are equal to 1 we recover the Cartesian version of these formulas.

Cylindrical polar coordinates:
In cylindrical polar coordinates \((\rho, \phi, z)\) with \((h_\rho, h_\phi, h_z) = (1, \rho, 1)\) we get
\[
\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho F_\rho \right) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z},
\]
\[
\nabla \times \mathbf{F} = \frac{1}{\rho} \left| \begin{array}{ccc}
\mathbf{e}_\rho & \rho \mathbf{e}_\phi & \mathbf{e}_z \\
\frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\
\rho F_\rho & \rho F_\phi & F_z
\end{array} \right|,
\]
\[
\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.
\]

Spherical polar coordinates:
In cylindrical polar coordinates \((r, \theta, \phi)\) with \((h_r, h_\theta, h_\phi) = (1, r, r \sin \theta)\) we get
\[
\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 F_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta F_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi},
\]
\[
\nabla \times \mathbf{F} = \frac{1}{r^2 \sin \theta} \left| \begin{array}{ccc}
\mathbf{e}_r & r \mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\
r F_r & r F_\theta & r \sin \theta F_\phi
\end{array} \right|,
\]
\[
\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}.
\]