VECTOR CALCULUS

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Introduction

The aim of this course is to extend your ideas about calculus to higher dimensions. We'll learn how differentiate functions of several variables, integrate over lines, surfaces and volume. We will be introduced to several different coordinate systems and learn how the coordinates, and small changes in them, relate to the position vectors they describe. Later in the course we'll look at solving some elliptic boundary value problems, which will put all our new knowledge of higher dimensional calculus into action. Finally, the last bit of the course will talk about tensors and why they're so useful.

On notation. Throughout this course a column vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is to be interpreted as the vector $\mathbf{x} = a\,\mathbf{e}_x + b\,\mathbf{e}_y + c\,\mathbf{e}_z$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are the basis vectors aligned with the fixed Cartesian x, y and z-axes in $\mathbf{R}^3$. We will often use the identification

$$\mathbf{x} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3,$$

i.e. $\mathbf{e}_2 \equiv \mathbf{e}_y$, $x_3 \equiv z$ etc, so that in suffix notation $\mathbf{x} = x_i\mathbf{e}_i$.

In preparing these lectures I have very much benefited from reading the notes of previous lecturers of the course: Ben Allanach, Stephen Cowley, Matthias Dörrzapf, Jonathan Evans & Robert Hunt. Comments and corrections very welcome at `acla2@cam.ac.uk`.

1. Differential geometry of curves

1.1. **Parametrised curves and arc-length.** A parametrised curve[1] C in $\mathbf{R}^3$ is just the image of a continuous map $\mathbf{x} : [a, b] \to \mathbf{R}^3$ in which $t \mapsto \mathbf{x}(t)$. We can express the curve in standard Cartesian coordinates

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_1(t) \\ x_3(t) \end{pmatrix} \equiv \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

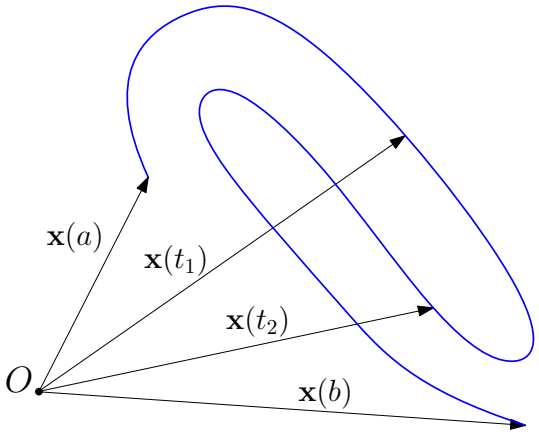

Figure 1. A curve in $\mathbf{R}^3$.

[1] In certain circles, the word *curve* is used for the map $t \mapsto \mathbf{x}(t)$ and the image of this mapping is then "the set of points on the curve". We're cavalier enough to glaze over this distinction.

We say such a curve is differentiable if each of the components $x_i(t)$ are differentiable functions of t and we say it is *regular* if $|\mathbf{x}'(t)| \neq 0$. We will call a parametrised curve C *smooth* if it is both differentiable and regular[2].

Recall that $x_i(t)$ is differentiable at the point t if, as $h \to 0$

$$x_i(t+h) = x_i(t) + hx_i'(t) + o(h), \quad i = 1, \ldots, n$$

where $x_i'(t)$ is defined as the derivative of $x_i(t)$. In terms of vectors

$$\mathbf{x}(t+h) = \mathbf{x}(t) + h\mathbf{x}'(t) + o(h),$$

where $\mathbf{x}'(t)$ is the vector with components $x_i'(t)$, $i = 1, 2, 3$.

How might we define the length of such a curve? We could try to first approximate the curve by a sequence of straight line segments. Define a partition $\mathcal{P}$ of the interval $[a, b]$

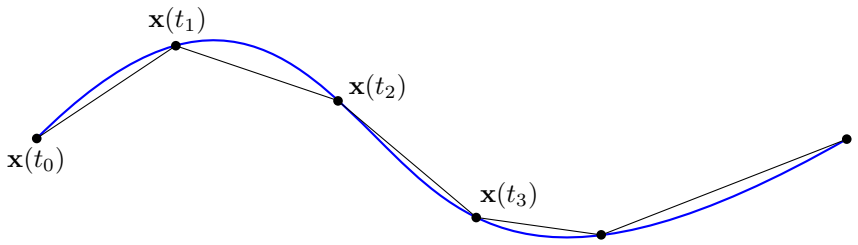

FIGURE 2. Partition of curve $\mathbf{x} = \mathbf{x}(t)$.

with $t_0 = a$ and $t_N = b$ and

$$t_0 < t_1 < \ldots < t_N,$$

and define $\Delta t = \max_i(t_{i+1} - t_i)$, i.e. the biggest gap between any of the t_i. Then we can define the length of C relative to $\mathcal{P}$ as

$$\ell(C, \mathcal{P}) = \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)|.$$

As Δt gets smaller, we expect $\ell(C, \mathcal{P})$ to become a better and better approximation to our intuitive notion of what the length of C would be. We define the length of C as

$$\ell(C) = \lim_{\Delta t \to 0} \ell(C, \mathcal{P}),$$

assuming the limit exists. If the limit doesn't exist, we say is a *non-rectifiable* curve. All the curves you meet in this course will be rectifiable.

Suppose that C is a differentiable curve and introduce a partition $\mathcal{P}$ defined by the points $\{t_i\}_{i=0}^N$ as before. Set $\Delta t_i = t_{i+1} - t_i$ so that $\Delta t = \max_i \Delta t_i$. We have

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i + \Delta t_i) = \mathbf{x}(t_i) + \mathbf{x}'(t_i)\Delta t_i + o(\Delta t_i),$$

hence

$$|\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)| = |\mathbf{x}'(t_i)|\Delta t_i + o(\Delta t_i).$$

[2]The regularity condition might seem odd, but consider planar curve $\mathbf{x}(t) = (t^2, t^3)$. Clearly $\mathbf{x}(t)$ is a differentiable function of t, but the curve exhibits a cusp at $t = 0$. Note also that $\mathbf{x}'(0) = 0$.

and we get a simple expression for the length of C relative to $\mathcal{P}$

$$\ell(C,\mathcal{P}) = \sum_{i=0}^{N-1} \Big(|\mathbf{x}'(t_i)|\Delta t_i + o(\Delta t_i) \Big).$$

Recall that $o(\Delta t_i)$ represents a function that, when divided by Δt_i, tends to zero as $\Delta t_i \to 0$. So for any $\epsilon > 0$, if $\Delta t = \max_i \Delta t_i$ is sufficiently small we have $|o(\Delta t_i)| \leq \epsilon \Delta t_i$ for each $i = 0, \ldots, N-1$. This gives the estimate

$$\left| \ell(C,\mathcal{P}) - \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)|\Delta t_i \right| = \left| \sum_{i=0}^{N-1} o(\Delta t_i) \right| \leq \epsilon \sum_{i=0}^{N-1} \Delta t_i = \epsilon(b-a).$$

This can be made as small as we please by choosing Δt smaller and smaller. So we find

$$\ell(C) = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)|\Delta t_i.$$

Obviously if $\Delta t \to 0$ then N, the number of terms in the sum, tends to infinity. You will prove in Analysis I that if $|\mathbf{x}'(t)|$ is continuous, say, then this expression converges to the Riemann integral

$$\int_a^b |\mathbf{x}'(t)| \, \mathrm{d}t.$$

So for a differentiable curve C parametrised by $t \mapsto \mathbf{x}(t)$, where $a \leq t \leq b$, we have

$$\ell(C) = \int_C \mathrm{d}s = \int_a^b |\mathbf{x}'(t)| \, \mathrm{d}t.$$

Similarly we define

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}(t))|\mathbf{x}'(t)| \, \mathrm{d}t.$$

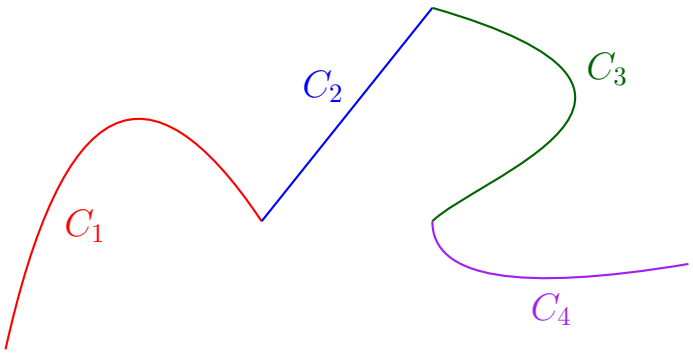

FIGURE 3. A piecewise smooth curve $C = C_1 + C_2 + C_3 + C_4$.

If a curve C is made up of M smooth curves $C_1, \ldots, C_M$ we say that C is *piecewise* smooth and write $C = C_1 + \cdots + C_M$, see Figure 3. All the curves you see in this course will be piecewise smooth. In this case we define

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \sum_{m=1}^{M} \int_{C_m} f(\mathbf{x}) \, \mathrm{d}s.$$

We think of $\mathrm{d}s$ as the arc-length element. Informally we have

$$\mathrm{d}s = |\mathbf{x}'(t)|\,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t, \quad \text{i.e.} \quad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Example. Let C be the circle with radius $r > 0$ in $\mathbf{R}^3$, parametrised by

$$\mathbf{x}(t) = (r\cos t, r\sin t, 0), \quad 0 \le t \le 2\pi.$$

Computing the derivatives of each component

$$\mathbf{x}'(t) = (-r\sin t, r\cos t, 0), \quad 0 \le t \le 2\pi.$$

The length of the curve C is defined by

$$\int_C \mathrm{d}s = \int_0^{2\pi} r\sqrt{\sin^2 t + \cos^2 t}\,\mathrm{d}t = 2\pi r.$$

We can also compute the integral

$$\int_C x^2 y\,\mathrm{d}s = \int_0^{2\pi} (r\cos t)^2 (r\sin t) r\sqrt{\sin^2 t + \cos^2 t}\,\mathrm{d}t = 0.$$

We have glossed over something subtle in the previous discussion: does $\ell(C)$ depend on the way in which we parametrise the curve C? Suppose we have two different parametrisations of the curve C

$$\mathbf{x} = \mathbf{x}_1(t), \quad a \le t \le b,$$
$$\mathbf{x} = \mathbf{x}_2(\tau), \quad \alpha \le \tau \le \beta,$$

where $\mathbf{x}_2(\tau) = \mathbf{x}_1(t(\tau))$ for some function $t = t(\tau)$. We must assume $\mathrm{d}t/\mathrm{d}\tau \neq 0$ so the relationship between t and τ is invertible and differentiable (see Analysis II). Note that

$$\mathbf{x}_2'(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_1(t(\tau)) = \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathbf{x}_1'(t(\tau)).$$

So from our definitions

$$\int_C f(\mathbf{x})\,\mathrm{d}s = \int_a^b f(\mathbf{x}_1(t))|\mathbf{x}_1'(t)|\,\mathrm{d}t.$$

Making the substitution $t = t(\tau)$ and assuming $\mathrm{d}t/\mathrm{d}\tau > 0$, the latter integral becomes

$$\int_\alpha^\beta f(\mathbf{x}_2(\tau))|\mathbf{x}_1'(t(\tau))|\frac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathrm{d}\tau = \int_\alpha^\beta f(\mathbf{x}_2(\tau))|\mathbf{x}_2'(\tau)|\,\mathrm{d}\tau$$

which is precisely what the integral $\int_C f\,\mathrm{d}s$ would have been if we'd used the parametrisation $\mathbf{x} = \mathbf{x}_2(\tau)$. So we get the same answer regardless of parametrisation – phew! The same result also holds if $\mathrm{d}t/\mathrm{d}\tau < 0$ (exercise).

Given there are lots of different choices of parametrisation, we might choose one that makes our life easy. Think back to the measurement of the length of a curve. We needn't measure the length of the entire curve – we could measure the length when we're part of the way along. Define the arc-length function

$$s(t) = \int_a^t |\mathbf{x}'(\tau)|\,\mathrm{d}\tau.$$

Note that $s(a) = 0$ and $s(b) = \ell(C)$, as expected. That the arc-length function is an increasing function of t since

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{x}'(t)| \geq 0.$$

In addition, if the curve is regular, i.e. $\mathbf{x}'(t) \neq 0$, then $s = s(t)$ is a strictly increasing function of t, so we can invert the relationship between t and s to find

$$t = t(s).$$

This allows us to parametrise a curve C using arc-length s. If we write $\mathbf{r}(s) = \mathbf{x}(t(s))$, then $s \mapsto \mathbf{r}(s)$ gives the curve C, where $0 \leq s \leq \ell(C)$. Using the chain rule

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\mathrm{d}s/\mathrm{d}t} = \frac{1}{|\mathbf{x}'(t(s)|}.$$

So we can compute

$$\mathbf{r}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}(t(s)) = \frac{\mathrm{d}t}{\mathrm{d}s}\mathbf{x}'(t(s)) = \frac{\mathbf{x}'(t(s))}{|\mathbf{x}'(t(s))|}.$$

In particular, $|\mathbf{r}'(s)| = 1$. This is consistent, since we then get

$$\int_0^{\ell(C)} |\mathbf{r}'(s)|\,\mathrm{d}s = \ell(C).$$

The vector $\mathbf{r}'(s)$ represents the unit tangent vector to the curve.

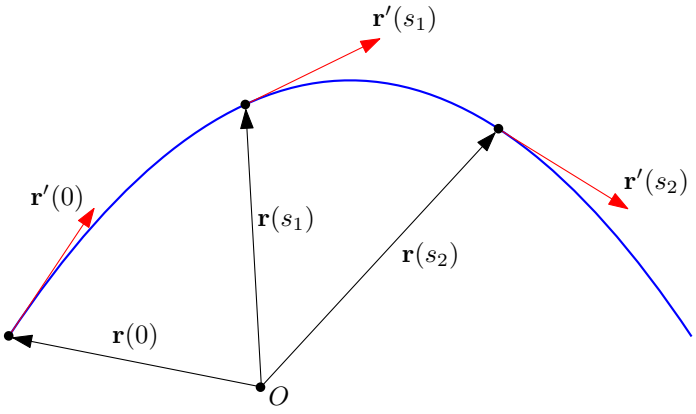

FIGURE 4. Some tangent vectors.

1.2. **Curvature and torsion.** Throughout this section we will talk about a generic smooth curve C parametrised by arc length $s \mapsto \mathbf{r}(s)$. We define the *tangent vector*

$$\mathbf{t}(s) = \mathbf{r}'(s).$$

We already know this is a unit vector. Since the length of $\mathbf{t}(s)$ doesn't change with s, the second derivative $\mathbf{r}''(s) = \mathbf{t}'(s)$ only measures how much the tangent vector changes direction. Intuitively, if $|\mathbf{r}''(s)|$ is large, it means the curve is rapidly changing direction, whereas if $|\mathbf{r}''(s)|$ is small then the curve is approximately flat. This leads us to the definition of *curvature*

$$\kappa(s) = |\mathbf{r}''(s)| = |\mathbf{t}'(s)|.$$

Since $\mathbf{t} = \mathbf{r}'(s)$ is a unit vector, differentiating $\mathbf{t} \cdot \mathbf{t} = 1$ gives $\mathbf{t} \cdot \mathbf{t}' = 0$. So we define the *principal normal* to the curve C by the formula

$$\mathbf{t}' = \kappa \mathbf{n}.$$

Note that $\mathbf{n}$ is a unit vector and is normal to C since $\mathbf{t} \cdot \mathbf{n} = 0$ everywhere. Given $\{\mathbf{t}, \mathbf{n}\}$ we can define an orthonormal basis by defining the *binormal* by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Since $|\mathbf{b}| = 1$ we have $\mathbf{b}' \cdot \mathbf{b} = 0$. Also, since $\mathbf{t} \cdot \mathbf{b} = 0$ and $\mathbf{n} \cdot \mathbf{b} = 0$ we have

$$(\mathbf{t} \cdot \mathbf{b})' = \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = \kappa \mathbf{n} \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = \mathbf{t} \cdot \mathbf{b}'.$$

So $\mathbf{b}'$ is perpendicular to both $\mathbf{b}$ and $\mathbf{t}$, i.e. $\mathbf{b}'$ must be parallel to $\mathbf{n}$. We define the *torsion* $\tau(s)$ by the equation

$$\mathbf{b}' = -\tau \mathbf{n}.$$

So we have the two equations

$$\mathbf{t}' = \kappa \mathbf{n}, \quad \mathbf{b}' = -\tau \mathbf{n}.$$

Proposition. *The curvature $\kappa(s)$ and torsion $\tau(s)$ determine a smooth curve upto translation and orientation.*

Proof. Since $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ we find the two coupled equations

$$\mathbf{t}' = \kappa(\mathbf{b} \times \mathbf{t}), \quad \mathbf{b}' = \tau(\mathbf{t} \times \mathbf{b}).$$

This is six equations for six unknowns. Given $\kappa(s)$, $\tau(s)$, $\mathbf{t}(0)$, $\mathbf{b}(0)$ we can construct $\mathbf{t}(s)$, $\mathbf{b}(s)$ and hence $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. Hence the result. ■

1.3. **Radius of curvature.** Let us Taylor expand a generic curve about the point $s = 0$. Writing $\mathbf{t}(0) = \mathbf{t}$ etc. we find

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{r}(0) + s\mathbf{r}'(0) + \tfrac{1}{2}s^2\mathbf{r}''(0) + o(s^2) \\ &= \mathbf{r} + s\mathbf{t} + \tfrac{1}{2}s^2\kappa\mathbf{n} + o(s^2). \end{aligned} \tag{$\star$}$$

Suppose we want to find a circle that is a "best fit" to the curve at $\mathbf{r} = \mathbf{r}(0)$. We assume without loss of generality we can assume $\mathbf{t}(0)$ is horizontal, so for the circle to be tangent to the curve at $\mathbf{r}(0)$, we will want the center of the circle directly above $\mathbf{r}(0)$ as in Figure 5. If the radius of this circle is r, then it can be described by the equation

$$\mathbf{x}(\theta) = \mathbf{r} + r(1 - \cos\theta)\mathbf{n} + r\sin\theta\mathbf{t}.$$

Expanding for small θ we find

$$\mathbf{x}(\theta) = \mathbf{r} + r\theta\mathbf{t} + \tfrac{1}{2}r\theta^2\mathbf{n} + o(\theta^2).$$

Measuring the arc-length of the circle from the point $\theta = 0$, we can write $s = r\theta$, so

$$\mathbf{x}(\theta) = \mathbf{r} + s\mathbf{t} + \tfrac{1}{2}s^2r^{-1}\mathbf{n} + o(s^2).$$

Comparing to $(\star)$ we find that the circle of best fit satisfies $\kappa = 1/r$. We define $r(s) = 1/\kappa(s)$ as the *radius of curvature* of $\mathbf{r} = \mathbf{r}(s)$.

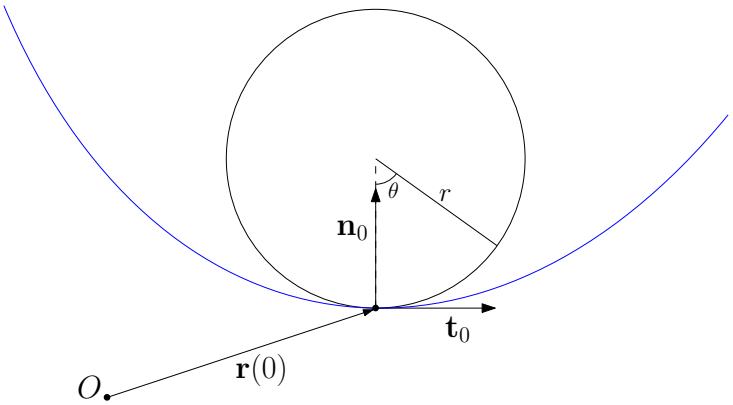

FIGURE 5. Radius of curvature.

1.4. **Gauss, pizza and curvature of surfaces (non-examinable).** Let S be a surface and at the point $\mathbf{x} \in S$ consider the normal $\mathbf{n} = \mathbf{n}(\mathbf{x})$. This defines a family of planes that contain $\mathbf{n}$ and intersect S. Each such intersection defines a curve, whose curvature κ we can measure at the point $\mathbf{x}$. Define the *Gaussian curvature* at $\mathbf{x}$ as

$$K = \kappa_{\min}\kappa_{\max}$$

where $\kappa_{\min}$ and $\kappa_{\max}$ are the minimum and maximum values of all the curvatures of the curves measured at $\mathbf{x}$. Gauss proved the following remarkable result

Theorem (Theorema Egregium – Remarkable Theorem!)**.** *The curvature K is intrinsic to the surface S, it does not depend on how you embed the surface in $\mathbf{R}^3$.*

In less abstract terms, it tells us that the Gaussian curvature of a surface does not change if you bend the surface without stretching it. For example, a flat piece of paper can be rolled up, without any stretching, to form a cylinder. Obviously for a flat piece of paper we have $K = 0$, so we deduce that for a cylinder $K = 0$ also.

As another example, consider the sphere of radius R. Any plane that is formed that contains a normal to the sphere produces a great circle of radius R. Remembering the link between radius of curvature and curvature of a curve, we deduce for a sphere of radius R

$$K = \frac{1}{R^2} > 0.$$

Gauss' theorem tells us, then, that there is no way of deforming a piece of paper onto a sphere, without stretching/crumpling it.

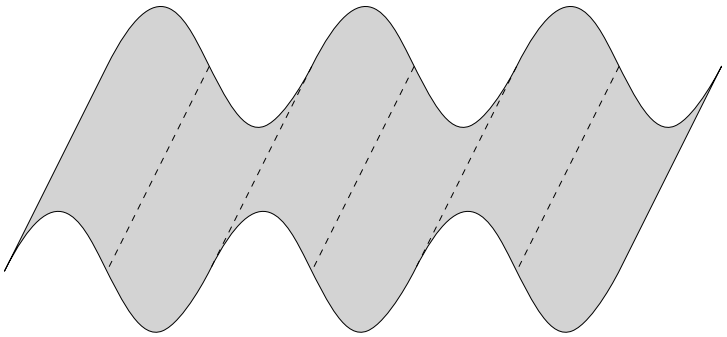

FIGURE 6. Corrugated iron, the dashed lines represent lines of zero curvature.

Yet another example is corrugated iron that you see on roofs. Note that at each point on a piece of corrugated iron you can form a straight line that runs along one of the axes of the sheet of metal. This means that at every point we have $\kappa_{\min} = 0$ and so $K = 0$ everywhere. This makes the iron rigid along this axis, because any deformation across it would mean that $K \neq 0$ there, and this can only happen if there is stretching within the material. Iron is reasonably tough to stretch, so we get rigidity.

Finally, Gauss' theorem helps you eat pizza. I will explain why in the lecture.

2. Coordinates, Differentials and Gradients

2.1. **Differentials and first order changes.** Recall the the *differential* $\mathrm{d}f$ of a function $f = f(u_1, \ldots, u_n)$ is defined, using summation convention, by

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\,\mathrm{d}u_i.$$

Here the $\{\mathrm{d}u_i\}$ are called differential forms. They are defined to be linearly independent, so $\alpha_i\,\mathrm{d}u_i = 0$ if and only if $\alpha_i = 0$ for each i. Similarly for $\mathbf{x} = \mathbf{x}(u_1, \ldots, u_n)$ a vector

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_i}\mathrm{d}u_i.$$

Example. Suppose $f(u, v, w) = u^2 + w\sin v$. Then

$$\mathrm{d}f = 2u\,\mathrm{d}u + w\cos v\,\mathrm{d}v + \sin v\,\mathrm{d}w.$$

Given the vector field

$$\mathbf{x}(u, v, w) = \begin{pmatrix} u^2 - v^2 \\ w \\ e^v \end{pmatrix}$$

with respect to the usual Cartesian axes, we compute

$$\mathrm{d}\mathbf{x} = \begin{pmatrix} 2u \\ 0 \\ 0 \end{pmatrix}\mathrm{d}u + \begin{pmatrix} -2v \\ 0 \\ e^v \end{pmatrix}\mathrm{d}v + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\mathrm{d}w.$$

Differentials encode information about how a function or vector changes when when we change the coordinates. Indeed, by ordinary calculus

$$f(u_1 + \delta u_1, \ldots, u_n + \delta u_n) - f(u_1, \ldots, u_n) = \frac{\partial f}{\partial u_i}\delta u_i + o(\delta\mathbf{u})$$

where $\delta\mathbf{u} = (\delta u_1, \ldots, \delta u_n)$ and $o(\delta\mathbf{u})$ means a function that obeys

$$\frac{o(\delta\mathbf{u})}{|\delta\mathbf{u}|} \to 0 \quad \text{as } |\delta\mathbf{u}| \to 0.$$

So if δf denotes the change in f after the change in coordinates

$$(u_1, \ldots, u_n) \mapsto (u_1 + \delta u_1, \ldots, u_n + \delta u_n)$$

then we have, to first order

$$\delta f \approx \frac{\partial f}{\partial u_i}\,\delta u_i,$$

and similarly for vectors

$$\delta\mathbf{x} \approx \frac{\partial \mathbf{x}}{\partial u_i}\,\delta u_i.$$

Differentials give us a rigorous way of discussing changes upto first order.

Another useful feature of differentials is that they give you the chain rule for free. Let $(u_1, \ldots, u_n)$ be a new set of coordinates, that depend on the old coordinates $(x_1, \ldots, x_n)$. If $F(u_1, \ldots, u_n) = f(x_1, \ldots, x_n)$ with $x_i = x_i(u_1, \ldots, u_n)$ for $i = 1, \ldots, n$, we compute both sides of the identity $\mathrm{d}F = \mathrm{d}f$.

$$\frac{\partial F}{\partial u_i}\mathrm{d}u_i = \mathrm{d}F = \mathrm{d}f = \frac{\partial f}{\partial x_j}\mathrm{d}x_j = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_i}\mathrm{d}u_i.$$

Since $\{\mathrm{d}u_1, \ldots, \mathrm{d}u_n\}$ form a basis we can identify the coefficients of $\mathrm{d}u_i$ to conclude

$$\frac{\partial F}{\partial u_i} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_i},$$

which is the chain rule for partial derivatives.

2.2. **Coordinates and line elements.** The idea behind a set of coordinates is that they give you a nice way of labelling points in space. You've seen at least two different sets of coordinates before: Cartesian coordinates (x, y) and polar coordinates (r, θ). They are related by the invertible[3] relationship

$$x = r\cos\theta, \quad y = r\sin\theta.$$

A general set of coordinates (u, v) on $\mathbf{R}^2$ can be defined using their relationship to the standard Cartesian coordinates (x, y), by specifying a pair of smooth functions

$$x = x(u, v), \quad y = y(u, v)$$

that can be inverted to give smooth functions $u = u(x, y)$ and $v = v(x, y)$. Since (x, y) denotes a unique point in $\mathbf{R}^2$, so does each (u, v), since all the relationships are invertible. So all a coordinate system does is give you some way of labelling points in space.

Similarly, in $\mathbf{R}^3$ we can use (u, v, w) as a set of coordinates defined by smooth functions

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w),$$

or simply $\mathbf{x} = \mathbf{x}(u, v, w)$ that can be inverted to give each of (u, v, w) in terms of (x, y, z).

Let us consider standard Cartesian coordinates

$$\mathbf{x}(x, y) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \equiv x\,\mathbf{e}_x + y\,\mathbf{e}_y.$$

The basis vectors $\{\mathbf{e}_x, \mathbf{e}_y\}$ are orthonormal and point in the same direction at every point in $\mathbf{R}^2$. The unit vector $\mathbf{e}_x$ points in the direction of changing x with y constant and the unit vector $\mathbf{e}_y$ points in the direction of changing y with x constant. Said differently,

$$\mathbf{e}_x = \frac{\partial_x \mathbf{x}(x, y)}{|\partial_x \mathbf{x}(x, y)|}, \quad \mathbf{e}_y = \frac{\partial_y \mathbf{x}(x, y)}{|\partial_y \mathbf{x}(x, y)|}.$$

where $\partial_x \equiv \partial/\partial x$ etc. The great thing about Cartesian coordinates is the following

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial x}\,\mathrm{d}x + \frac{\partial \mathbf{x}}{\partial y}\,\mathrm{d}y = \mathrm{d}x\,\mathbf{e}_x + \mathrm{d}y\,\mathbf{e}_y,$$

i.e. the change *in the coordinate* $x \mapsto x + \delta x$ gives a first order (actually exact in this case) change *to the vector* $\mathbf{x} \mapsto \mathbf{x} + \delta x\,\mathbf{e}_x$. This is certainly not the case with all coordinate systems. We call $\mathrm{d}\mathbf{x}$ the line element, it tells us how small changes in the coordinates produce small changes in the vector.

Using polar coordinates (r, θ) so that

$$\mathbf{x}(r, \theta) = \left(\begin{smallmatrix} r\cos\theta \\ r\sin\theta \end{smallmatrix}\right) \equiv r\mathbf{e}_r.$$

In this case we have the basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta\}$, where

$$\mathbf{e}_r = \left(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right), \quad \mathbf{e}_\theta = \left(\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix}\right).$$

Again $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ form an orthonormal basis for each (r, θ), but not the same basis – the vectors change with position. The associated basis vectors are obtained from the coordinate description of $\mathbf{x} = \mathbf{x}(r, \theta)$ via

$$\mathbf{e}_r = \frac{\partial_r \mathbf{x}(r, \theta)}{|\partial_r \mathbf{x}(r, \theta)|}, \quad \mathbf{e}_\theta = \frac{\partial_\theta \mathbf{x}(r, \theta)}{|\partial_\theta \mathbf{x}(r, \theta)|}.$$

[3]Well, not quite – polar coordinates are badly behaved at the origin, but this doesn't really cause too much of an issue in practice.

Since $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthogonal, we call (r, θ) an *orthogonal set of curvilinear coordinates*. For polar coordinates, using $\partial \mathbf{e}_r/\partial\theta = \mathbf{e}_\theta$ and $\partial \mathbf{e}_r/\partial r = 0$ we find the line element

$$\mathrm{d}\mathbf{x} = \frac{\partial}{\partial r}(r\mathbf{e}_r)\,\mathrm{d}r + \frac{\partial}{\partial\theta}(r\mathbf{e}_r)\,\mathrm{d}\theta = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta.$$

We see that a small change $\theta \mapsto \theta + \delta\theta$ does *not* give a first order change of $\mathbf{x}$ by $\delta\theta\,\mathbf{e}_\theta$. In this case the first order change would be $\mathbf{x} \mapsto \mathbf{x} + r\,\delta\theta\,\mathbf{e}_\theta$.

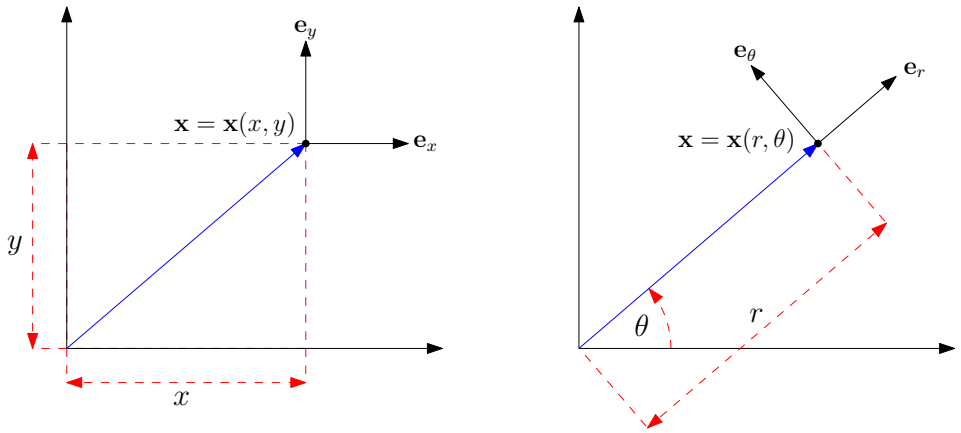

FIGURE 7. Position vector $\mathbf{x}$ described using Cartesian ($\mathbf{x} = \mathbf{x}(x, y)$, left) and plane polar ($\mathbf{x} = \mathbf{x}(r, \theta)$, right) coordinates.

2.2.1. *Orthogonal curvilinear coordinates.* In general we say that (u, v, w) are a set of *orthogonal curvilinear coordinates* if the vectors

$$\mathbf{e}_u = \frac{\partial_u \mathbf{x}}{|\partial_u \mathbf{x}|}, \quad \mathbf{e}_v = \frac{\partial_v \mathbf{x}}{|\partial_v \mathbf{x}|}, \quad \mathbf{e}_w = \frac{\partial_w \mathbf{x}}{|\partial_w \mathbf{x}|}, \quad \left(\partial_u \equiv \frac{\partial}{\partial u} \text{ etc.}\right)$$

form a right-handed, orthonormal basis. This means they are mutually orthogonal unit vectors and $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$. Just like polar coordinates, these vectors $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ form a basis at any point in $\mathbf{R}^3$, but not necessarily the same basis at each point.

It is standard practice to write $h_u = |\partial_u \mathbf{x}|$, so that $\partial_u \mathbf{x} = h_u \mathbf{e}_u$ and so on. These are called *scale factors*. This gives a simple expression for the line element

$$\begin{aligned}\mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial u}\,\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\,\mathrm{d}v + \frac{\partial \mathbf{x}}{\partial w}\,\mathrm{d}w \\ &= h_u\,\mathrm{d}u\,\mathbf{e}_u + h_v\,\mathrm{d}v\,\mathbf{e}_v + h_w\,\mathrm{d}w\,\mathbf{e}_w.\end{aligned}$$

You can see why $\{h_u, h_v, h_w\}$ are called *scale factors*: they tell you how small changes in the coordinates (u, v, w) scale up to first order changes in $\mathbf{x}$ in the directions $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$.

2.2.2. *Cylindrical polar coordinates.* Cylindrical polar coordinates (ρ, ϕ, z) are defined by

$$\mathbf{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix} \quad 0 \le \rho < \infty, \quad 0 \le \phi < 2\pi, \quad -\infty < z < \infty.$$

These are orthogonal curvilinear coordinates with

$$\mathbf{e}_\rho = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The scale factors are $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$. The corresponding line element is

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho\,\mathbf{e}_\rho + \rho\,\mathrm{d}\phi\,\mathbf{e}_\phi + \mathrm{d}z\,\mathbf{e}_z.$$

In particular, this tells us that the change in the coordinate $\phi \mapsto \phi + \delta\phi$ produces a first order change $\mathbf{x} \mapsto \mathbf{x} + \rho\,\delta\phi\,\mathbf{e}_\phi$ and *not* simply $\mathbf{x} + \delta\phi\,\mathbf{e}_\phi$. In cylindrical polars

$$\mathbf{x} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix} = \rho \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \rho\mathbf{e}_\rho + z\mathbf{e}_z.$$

See Handout 1 for a picture.

2.2.3. *Spherical polar coordinates.* Spherical polar coordinates (r, θ, ϕ) are defined by

$$\mathbf{x}(r,\theta,\phi) = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix} \quad 0 \le r < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi.$$

These are orthogonal curvilinear coordinates with

$$\mathbf{e}_r = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

The scale factors are $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. The corresponding line element is

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi.$$

In this case we see that a change in the (azimuthal) coordinate $\phi \mapsto \phi + \delta\phi$ produces a first order change $\mathbf{x} \mapsto \mathbf{x} + r\sin\theta\,\delta\phi\,\mathbf{e}_\phi$. In spherical polars

$$\mathbf{x} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix} = r \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} = r\mathbf{e}_r.$$

See Handout 1 for a picture.

2.3. **The gradient operator.** For a function scalar function $f : \mathbf{R}^3 \to \mathbf{R}$ we define the *gradient* of f, labelled ∇f, by

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \mathbf{h} \cdot \nabla f(\mathbf{x}) + o(\mathbf{h})$$

where $o(\mathbf{h})$ means a function that obeys

$$\frac{o(\mathbf{h})}{|\mathbf{h}|} \to 0 \quad \text{as } |\mathbf{h}| \to 0.$$

Note that we couldn't use a kind of difference-quotient for this definition, because we're dealing with vectors. The difference-quotient approach would have produced terms like

$$? \ \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} \ ?$$

which, I hope, fills the reader with a sense dread.

We can also interpret ∇f in terms of directional derivatives. The directional derivative of f in the direction $\mathbf{v}$, denoted $D_\mathbf{v} f$ or $\partial f/\partial \mathbf{v}$, is defined by

$$D_\mathbf{v} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{x})}{t}$$

or equivalently, in terms of our more grown-up language

$$f(\mathbf{x} + t\mathbf{v}) = f(\mathbf{x}) + tD_\mathbf{v} f(\mathbf{x}) + o(t).$$

Using $\mathbf{h} = t\mathbf{v}$ in our definition of ∇f we find

$$f(\mathbf{x} + t\mathbf{v}) = f(\mathbf{x}) + t\mathbf{v} \cdot \nabla f(\mathbf{x}) + o(t)$$

from which we deduce

$$D_\mathbf{v} f = \mathbf{v} \cdot \nabla f.$$

We know from the Cauchy-Schwarz inequality that $\mathbf{a} \cdot \mathbf{b}$ is maximised when $\mathbf{a}$ points in the same direction as $\mathbf{b}$. We arrive at the useful interpretation:

$$\nabla f \text{ corresponds to the direction of greatest increase of } f.$$

And by exactly the same reasoning

$$-\nabla f \text{ corresponds to the direction of greatest decrease of } f.$$

Example. Suppose $f(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$. Then we compute

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}) &= \tfrac{1}{2}(\mathbf{x}+\mathbf{h}) \cdot (\mathbf{x}+\mathbf{h}) \\ &= \tfrac{1}{2}|\mathbf{x}|^2 + \tfrac{1}{2}\left(2\mathbf{h} \cdot \mathbf{x}\right) + \tfrac{1}{2}|\mathbf{h}|^2 \\ &= f(\mathbf{x}) + \mathbf{h} \cdot \mathbf{x} + o(\mathbf{h}). \end{aligned}$$

So $\nabla f(\mathbf{x}) = \mathbf{x}$.

Suppose we have a curve $t \mapsto \mathbf{x}(t)$. How does $F(t) = f(\mathbf{x}(t))$ change with t. Fix t use the notation $\delta\mathbf{x} = \mathbf{x}(t+\delta t) - \mathbf{x}(t)$. Then

$$\begin{aligned} F(t+\delta t) &= f(\mathbf{x}(t+\delta t)) \\ &= f(\mathbf{x}(t) + \delta\mathbf{x}) \\ &= f(\mathbf{x}(t)) + \delta\mathbf{x} \cdot \nabla f(\mathbf{x}(t)) + o(\delta\mathbf{x}). \end{aligned}$$

Since $\delta\mathbf{x} = \mathbf{x}'(t)\delta t + o(\delta t)$ and $F(t) = f(\mathbf{x}(t))$ we find

$$F(t+\delta t) = F(t) + \mathbf{x}'(t) \cdot \nabla f(\mathbf{x}(t))\delta t + o(\delta t).$$

This gives

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \nabla f(\mathbf{x}(t)).$$

Finally, consider a two dimensional surface in $\mathbf{R}^3$ defined implicitly by

$$f(\mathbf{x}) = 0.$$

Suppose $t \mapsto \mathbf{x}(t)$ is any curve in this surface so that $f(\mathbf{x}(t)) = 0$ holds identically. Then by the previous result

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) = \mathbf{x}'(t) \cdot \nabla f(\mathbf{x}(t)).$$

So the vector ∇f is perpendicular to the tangent vector $\mathbf{x}'(t)$. Since this is true for any curve in the surface, we conclude that $\nabla f(\mathbf{x})$ is *normal* to the surface at the point $\mathbf{x}$.

2.4. **Computing the gradient.** How do we actually compute ∇f in practice? Our definition is a little abstract, because it doesn't mention what coordinates we're using to describe the vectors $\mathbf{x}$ and $\mathbf{h}$. In general this might be difficult, because we'd have to figure out how to change our coordinates (u, v, w) in such a way to get exactly $\mathbf{x} + \mathbf{h}$.

In Cartesian coordinates life is simple because to induce a change by $\delta\mathbf{x}$, we can just change each of the coordinates $x \mapsto x + \delta x$ etc, so[4]

$$f(\mathbf{x} + \delta\mathbf{x}) = f(x + \delta x, y + \delta y, x + \delta z).$$

[4]Warning: universal abuse of notation incoming! What we *should* be writing here is $F(x, y, z) = f(\mathbf{x}(x, y, z))$, where F is a function of the coordinates and f a function of the position vector. In Cartesian coordinates this might seem excessive, but in other coordinate systems $F(u, v, w) = f(\mathbf{x}(u, v, w))$ would be more proper. However, this becomes cumbersome because we'd have to use a different F each time we used a different coordinate system, whilst possibly referring to the same f. E.g. if $f(\mathbf{x}) = |\mathbf{x}|^2$ then we

By the ordinary rules of calculus then

$$\begin{aligned} f(\mathbf{x}+\delta\mathbf{x}) &= f(x+\delta x, y+\delta y, z+\delta z) \\ &= f(\mathbf{x}) + \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial z}\,\delta z + o(\delta\mathbf{x}) \\ &= f(\mathbf{x}) + \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} \cdot \delta\mathbf{x} + o(\delta\mathbf{x}). \end{aligned}$$

So according to our definition of the gradient

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}.$$

Using suffix notation

$$\nabla f = \mathbf{e}_i \frac{\partial f}{\partial x_i} \quad \text{or} \quad [\nabla f]_i = \frac{\partial f}{\partial x_i}.$$

This should be familiar to you from IA differential equations. You can consider the gradient as a vector differential operator. We've just found that in Cartesian coordinates

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}.$$

Example. Let $f = \frac{1}{2}(x^2+y^2+z^2) = \frac{1}{2}|\mathbf{x}|^2$ as before. We compute the gradient using Cartesian coordinates and suffix notation. Since $[\nabla f]_i = \partial f/\partial x_i$ we have

$$[\nabla f]_i = \frac{1}{2}\frac{\partial}{\partial x_i}(x_j x_j) = \frac{1}{2}\left(\delta_{ij}x_j + x_j\delta_{ij}\right) = x_i,$$

so $\nabla f = x_i\mathbf{e}_i = \mathbf{x}$, as expected.

Recall the line element in Cartesian coordinates

$$\mathrm{d}\mathbf{x} = \mathrm{d}x\,\mathbf{e}_x + \mathrm{d}y\,\mathbf{e}_y + \mathrm{d}z\,\mathbf{e}_z \equiv \mathrm{d}x_i\,\mathbf{e}_i.$$

Also we have the differential

$$\mathrm{d}f = \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

Using both these expressions we find

$$\nabla f \cdot \mathrm{d}\mathbf{x} = \left(\mathbf{e}_i \frac{\partial f}{\partial x_i}\right) \cdot (\mathbf{e}_j\,\mathrm{d}x_j) = \frac{\partial f}{\partial x_i}(\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathrm{d}x_j = \frac{\partial f}{\partial x_i}\delta_{ij}\,\mathrm{d}x_j = \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i = \mathrm{d}f.$$

This statement is coordinate independent. In addition, computing $\mathrm{d}\mathbf{x}$ and $\mathrm{d}f$ in any coordinate system is easy. So we find a better definition of the gradient which will work in any coordinate system[5]

$$\boxed{\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{x}}$$

---

have $f(\mathbf{x}) = F(\rho,\phi,z) = \rho^2+z^2$ and $f(\mathbf{x}) = G(r,\theta,\phi) = r^2$ in cylindrical and spherical polars respectively. This quickly becomes cumbersome, so it is easier to just refer to everything as f.

[5]Again, there's a very subtle abuse of notation here. What I should have done was to define $F(x,y,z) = f(\mathbf{x}(x,y,z))$, then we would have arrived at $\mathrm{d}F = \nabla f\cdot\mathrm{d}\mathbf{x}$. In the general case with orthogonal curvilinear coordinates (u,v,w) we should write $F(u,v,w) = f(\mathbf{x}(u,v,w))$. In this setting the general result is

$$\nabla f = \frac{1}{h_u}\frac{\partial F}{\partial u}\,\mathbf{e}_u + \frac{1}{h_v}\frac{\partial F}{\partial v}\,\mathbf{e}_v + \frac{1}{h_w}\frac{\partial F}{\partial w}\,\mathbf{e}_w$$

The gradient then falls on *the function of position* $f = f(\mathbf{x})$, whilst the derivatives fall on *the function of the coordinates* $F = F(u,v,w)$. This becomes cumbersome after a while, so we use f everywhere.

Proposition. *If (u, v, w) is a set of orthogonal curvilinear coordinates and $f = f(u, v, w)$*

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\,\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\,\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\,\mathbf{e}_w.$$

Proof. We know that if $f = f(u, v, w)$ and $\mathbf{x} = \mathbf{x}(u, v, w)$ then

$$\mathrm{d}f = \frac{\partial f}{\partial u}\,\mathrm{d}u + \frac{\partial f}{\partial v}\,\mathrm{d}v + \frac{\partial f}{\partial w}\,\mathrm{d}w, \quad \mathrm{d}\mathbf{x} = h_u\,\mathrm{d}u\,\mathbf{e}_u + h_v\,\mathrm{d}v\,\mathbf{e}_v + h_w\,\mathrm{d}w\,\mathbf{e}_w$$

where $h_u = |\partial_u \mathbf{x}|$ etc. By writing

$$\nabla f = (\nabla f)_u\,\mathbf{e}_u + (\nabla f)_v\,\mathbf{e}_v + (\nabla f)_w\,\mathbf{e}_w$$

and substituting everything in, using the fact that $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ are orthonormal, we find

$$\frac{\partial f}{\partial u}\,\mathrm{d}u + \frac{\partial f}{\partial v}\,\mathrm{d}v + \frac{\partial f}{\partial w}\,\mathrm{d}w = h_u(\nabla f)_u\,\mathrm{d}u + h_v(\nabla f)_v\,\mathrm{d}v + h_w(\nabla f)_w\,\mathrm{d}w.$$

Using the linear independent of the differentials $\{\mathrm{d}u, \mathrm{d}v, \mathrm{d}w\}$ we find

$$(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u}, \quad (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v}, \quad (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w},$$

which is the required result. ∎

In cylindrical polar coordinates (ρ, ϕ, z) we have scale factors $h_\rho = 1, h_\phi = \rho, h_z = 1$. So if $f = f(\rho, \phi, z)$ we have

$$\nabla f = \frac{\partial f}{\partial \rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi + \frac{\partial f}{\partial z}\,\mathbf{e}_z.$$

In spherical polar coordinates (r, θ, ϕ) we have scale factors $h_r = 1, h_\theta = r, h_\phi = r\sin\theta$. So if $f = f(r, \theta, \phi)$ we have

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi.$$

Example. Consider the function $f(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$. Then

$$f = \begin{cases} \frac{1}{2}(x^2 + y^2 + z^2), & \text{in Cartesian coordinates,} \\ \frac{1}{2}(\rho^2 + z^2), & \text{in cylindrical polar coordinates,} \\ \frac{1}{2}r^2, & \text{in spherical polar coordinates.} \end{cases}$$

Using the appropriate definition of ∇ in each coordinate system

$$\nabla f = \begin{cases} x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z, & \text{in Cartesian coordinates,} \\ \rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z, & \text{in cylindrical polar coordinates,} \\ r\,\mathbf{e}_r, & \text{in spherical polar coordinates.} \end{cases}$$

Of course, all these answers are exactly the same vector $\mathbf{x}$, i.e. $\nabla(\frac{1}{2}|\mathbf{x}|^2) = \mathbf{x}$.

2.5. **Summary.** The following table gives a summary of our results for the most common coordinate systems used.

Cartesian	Cylindrical Polars	Spherical Polars
(x, y, z)	(ρ, ϕ, z)	(r, θ, ϕ)
$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\mathbf{x} = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$	$\mathbf{x} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$
$\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$	$\{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\}$	$\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$
$\mathbf{x} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$	$\mathbf{x} = \rho\,\mathbf{e}_\rho + z\,\mathbf{e}_z$	$\mathbf{x} = r\,\mathbf{e}_r$
$\mathrm{d}\mathbf{x} = \mathrm{d}x\,\mathbf{e}_x + \mathrm{d}y\,\mathbf{e}_y + \mathrm{d}z\,\mathbf{e}_z$	$\mathrm{d}\mathbf{x} = \mathrm{d}\rho\,\mathbf{e}_\rho + \rho\,\mathrm{d}\phi\,\mathbf{e}_\phi + \mathrm{d}z\,\mathbf{e}_z$	$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi$
$\nabla = \mathbf{e}_x\,\frac{\partial}{\partial x} + \mathbf{e}_y\,\frac{\partial}{\partial y} + \mathbf{e}_z\,\frac{\partial}{\partial z}$	$\nabla = \mathbf{e}_\rho\,\frac{\partial}{\partial \rho} + \mathbf{e}_\phi\,\frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{e}_z\,\frac{\partial}{\partial z}$	$\nabla = \mathbf{e}_r\,\frac{\partial}{\partial r} + \mathbf{e}_\theta\,\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi\,\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

3. Integration over lines, surfaces and volumes

3.1. **Line integrals.** For a vector field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ and a piecewise smooth curve C parametrised by $[a, b] \ni t \mapsto \mathbf{x}(t)$ we define the line integral

$$\int_C \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\, \mathrm{d}t.$$

We refer to $\mathrm{d}\mathbf{x} = \mathbf{x}'(t)\, \mathrm{d}t$ as the line element on C. Direction is important here: we assume the path is taken from $\mathbf{x}(a)$ to $\mathbf{x}(b)$. If we were traversing C in the opposite direction, the limits would be interchanged and we would write $\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ instead.

If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ represents the some force at the point $\mathbf{x}$, then we can interpret $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ as the work done by a particle moving along the path C, i.e. the limit of a sum of the form

$$\sum_i \mathbf{F}(\mathbf{x}_i) \cdot \Delta\mathbf{x}_i,$$

where $\Delta\mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$ and $|\Delta\mathbf{x}| = \max |\Delta\mathbf{x}_i|$ is small. The limit is then taken as $|\Delta\mathbf{x}| = \max_i |\Delta\mathbf{x}_i|$ tends to zero, c.f. definition of $\int_C f\, \mathrm{d}s$ in §1.

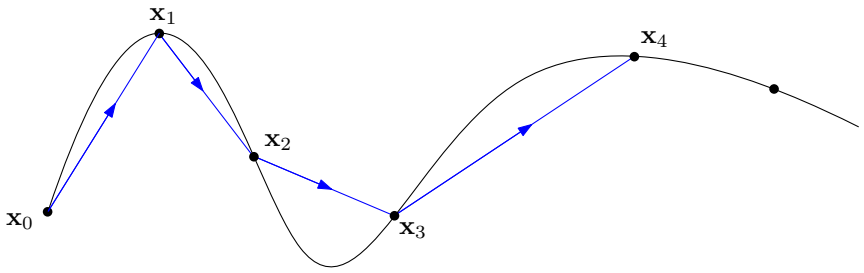

FIGURE 8. Line integrals as a limit of sum $\sum \mathbf{F}(\mathbf{x}_i) \cdot \Delta\mathbf{x}_i$.

Example. Consider the vector field given in Cartesian coordinates by

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x^2 y \\ yz \\ 2zx \end{pmatrix}.$$

For $t \in [0, 1]$ consider the two paths joining the origin to $(1, 1, 1)^t$

$$C_1 : t \mapsto \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \quad C_2 : t \mapsto \begin{pmatrix} t \\ t \\ t^2 \end{pmatrix}.$$

Then we have

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \begin{pmatrix} t^3 \\ t^2 \\ 2t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{d}t = \left(\frac{1}{4}t^4 + \frac{1}{3}t^3 + \frac{2}{3}t^3 \right) \Big|_0^1 = \frac{5}{4},$$

and

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \begin{pmatrix} t^3 \\ t^3 \\ 2t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2t \end{pmatrix} \mathrm{d}t = \left(\frac{1}{4}t^4 + \frac{1}{4}t^4 + \frac{4}{5}t^5 \right) \Big|_0^1 = \frac{13}{10}.$$

So in general, the line integral depends on the path taken between two points.

Example. A particle at the point $\mathbf{x}$ experiences a force given in cylindrical polars by

$$\mathbf{F}(\mathbf{x}) = z\rho\, \mathbf{e}_\phi.$$

Calculate the work done by travelling along the path

$$C : [0, 2\pi] \ni t \mapsto \begin{pmatrix} a\cos t \\ a\sin t \\ t \end{pmatrix}.$$

We can do the calculation in cylindrical polars. Recall the line element

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho\,\mathbf{e}_\rho + \rho\,\mathrm{d}\phi\,\mathbf{e}_\phi + \mathrm{d}z\,\mathbf{e}_z.$$

So $\mathbf{F}\cdot\mathrm{d}\mathbf{x} = z\rho^2\,\mathrm{d}\phi$. On the path we see that

$$(\rho,\phi,z) = (a,t,t) \quad \Rightarrow \quad (\mathrm{d}\rho,\mathrm{d}\phi,\mathrm{d}z) = (0,\mathrm{d}t,\mathrm{d}t),$$

so we find $\mathbf{F}\cdot\mathrm{d}\mathbf{x} = a^2 t\,\mathrm{d}t$. Hence

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x} = a^2\int_0^{2\pi} t\,\mathrm{d}t = 2\pi^2 a^2.$$

If the curve $t\mapsto\mathbf{x}(t)$ is such that $\mathbf{x}(a)=\mathbf{x}(b)$, we say the curve is closed. In this case it is customary to write

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{x}$$

to indicate that the integral is being taken around a closed loop. This is sometimes referred to as the *circulation* of $\mathbf{F}$ around the loop C.

Example. Consider the previous example with $C = C_1 - C_2$, where the minus sign indicates we travel in the reverse direction along C_2. Then C is a closed curve and

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{x} = \int_{C_1}\mathbf{F}\cdot\mathrm{d}\mathbf{x} - \int_{C_2}\mathbf{F}\cdot\mathrm{d}\mathbf{x} = \frac{5}{4} - \frac{83}{60} = -\frac{2}{15}.$$

3.2. **Conservative forces and exact differentials.** In the previous section we saw how to interpret expressions of the form

$$\mathbf{F}\cdot\mathrm{d}\mathbf{x}$$

as long as they were inside an integral. This is another example of a *differential form*. We say $\mathbf{F}\cdot\mathrm{d}\mathbf{x}$ is *exact* if it can be written as the differential of a scalar function, i.e. $\mathbf{F}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}f$ for some function f. Recall the coordinate free definition of the gradient operator

$$\mathrm{d}f = \nabla f\cdot\mathrm{d}\mathbf{x}.$$

So $\mathbf{F}\cdot\mathrm{d}\mathbf{x}$ is exact if and only if $\mathbf{F} = \nabla f$ for some scalar function f. We call a vector field of the form $\mathbf{F} = \nabla f$ *conservative*. So we have the identification

$$\mathbf{F}\cdot\mathrm{d}\mathbf{x} \text{ is exact} \quad \Leftrightarrow \quad \mathbf{F} \text{ is conservative.}$$

Note the following properties of the differential: if λ,μ are constants and f,g are functions then

$$\mathrm{d}(\lambda f + \mu g) = \lambda\,\mathrm{d}f + \mu\,\mathrm{d}g,$$

and

$$\mathrm{d}(fg) = f(\mathrm{d}g) + g(\mathrm{d}f).$$

Using these rules it is often easy to spot if a a given differential $\mathbf{F}\cdot\mathrm{d}\mathbf{x}$ is exact.

Proposition. *If θ is exact then*

$$\oint_C \theta = 0$$

for any closed curve C.

Proof. If $\theta = \nabla f \cdot \mathrm{d}\mathbf{x}$ then using our definitions

$$\oint_C \theta = \oint \nabla f \cdot \mathrm{d}\mathbf{x} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\,\mathrm{d}t = f(\mathbf{x}(b)) - f(\mathbf{x}(a)),$$

and if the curve is closed then $\mathbf{x}(a) = \mathbf{x}(b)$, so we arrive at the result. ∎

From this we deduce that if $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is exact, or equivalently if $\mathbf{F}$ is conservative, then the line integral between two points $A = \mathbf{x}(a)$ and $B = \mathbf{x}(b)$ doesn't depend on the path taken. Indeed, if C_1 and C_2 are two different paths from A to B, consider the closed path $C = C_1 - C_2$, i.e. first traverse the paths along from A to B along C_1, then then travel back to A by going backwards along C_2. Then

$$0 = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}.$$

If $\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$, then this implies

$$\int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_A^B \mathrm{d}f = f(B) - f(A)$$

independent of path.

Let $(u_1, u_2, u_3) \equiv (u, v, w)$ be an arbitrary set of coordinates and let

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \theta = A(u, v, w)\,\mathrm{d}u + B(u, v, w)\,\mathrm{d}v + C(u, v, w)\,\mathrm{d}w \equiv \theta_i\,\mathrm{d}u_i.$$

A *necessary* condition for θ to be exact is

$$(\dagger) \qquad \frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}.$$

We say θ is *closed* if it obeys this condition. This necessary conditions comes from the following observation: if θ is exact then $\theta_i = \partial f / \partial u_i$ for some f and the constraint comes from using the symmetry of mixed partial derivatives. If the domain on which these differentials are defined is simply connected, then these conditions are also sufficient[6].

Example. (a) Given the differential $\theta = y\,\mathrm{d}x - x\,\mathrm{d}y$. Is it exact? Check necessary condition:

$$\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial x}(-x) = 1 + 1 \neq 0.$$

So θ is not exact.

(b) Compute the line integral

$$\oint_C 3x^2 y\,\mathrm{d}x + x^3\,\mathrm{d}y$$

where C is a really complicated looking loop. Answer: note that if $f(x, y) = x^3 y$ then the integral is equivalent to $\oint_C \mathrm{d}f$, so the integral is zero, no matter how complicated C is.

[6]This is remarkable. Asking if a form is exact is a question of analysis: does there exist a function for which $\theta = \mathrm{d}f$. However, it turns out that the sufficiency of our condition depends on the *topology* of our domain, in this case simple connectedness. So there seems to be some interesting back and forth between standard analysis of functions and the topology of their domain. There is a fascinating area of pure mathematics that fleshes out this strange link, called *cohomology*, and in particular *de Rham cohomology*.

Example. If a particle of mass m experiences a force $\mathbf{F}$ at the point $\mathbf{x}$ as it travels along a curve C then

$$\text{Work done} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = m \int_a^b \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}} \, \mathrm{d}t = \frac{1}{2} m |\dot{\mathbf{x}}|^2 \Big|_a^b,$$

which corresponds to the change in kinetic energy. If $\mathbf{F} = -\nabla V$, i.e. $\mathbf{F}$ is conservative, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = -\int_C \nabla V \cdot \mathrm{d}\mathbf{x} = V(\mathbf{x}(a)) - V(\mathbf{x}(b)).$$

Hence

$$\left(\frac{1}{2} m |\dot{\mathbf{x}}(t)|^2 + V(\mathbf{x}(t)) \right) \Big|_{t=a} = \left(\frac{1}{2} m |\dot{\mathbf{x}}(t)|^2 + V(\mathbf{x}(t)) \right) \Big|_{t=b}$$

meaning energy is *conserved*, where $V = V(\mathbf{x})$ is the potential energy.

3.3. **Integration in $\mathbf{R}^2$.** We want to integrate over a bounded region $D \subset \mathbf{R}^2$. The most obvious thing to do would be to first cover D with small, disjoint subsets A_{ij} with areas δA_{ij}, each of which contained inside a disc of radius ϵ. Then introduce the points (x_i, y_j) each contained within each A_{ij}. We then define

$$\int_D f(\mathbf{x}) \, \mathrm{d}A = \lim_{\epsilon \to 0} \sum_{i,j} f(x_i, y_j) \, \delta A_{ij}.$$

The integral exists if the limit is independent of choice of A_{ij} and (x_i, y_j).

This definition allows us to choose the shape of the A_{ij} and the most obvious choice would be to use rectangles with $\delta A_{ij} = \delta x \, \delta y$. We sum over subsets in a narrow horizontal strip of width δy, with y held constant. Then in taking the limit $\delta x \to 0$ we get

$$\delta y \left(\int_{X_y} f(x, y) \, \mathrm{d}x \right),$$

where $X_y = \{x : (x, y) \in D\}$. Then we sum over all such horizontal strips and take the limit $\delta y \to 0$. This gives

$$(\dagger) \qquad \int_D f(x, y) \, \mathrm{d}A = \int_Y \left(\int_{X_y} f(x, y) \, \mathrm{d}x \right) \mathrm{d}y.$$

Alternatively, if we sum over vertical strips first, we would obtain

$$(\dagger\dagger) \qquad \int_X \left(\int_{Y_x} f(x, y) \, \mathrm{d}y \right) \mathrm{d}x,$$

where $Y_x = \{y : (x, y) \in D\}$. See handout 2. It is a *theorem*, called Fubini's theorem, that if $\int_D f \, \mathrm{d}A$ exists, then it it coincides with both (†) and (††). You can prove this theorem in Part II Probability and Measure[7].

In summary, we have found that in Cartesian coordinates, the area element $\mathrm{d}A$ satisfies

$$\mathrm{d}A = \mathrm{d}x \, \mathrm{d}y = \mathrm{d}y \, \mathrm{d}x.$$

Example. Let D be a triangle with vertices $(0, 0), (1, 0), (0, 1)$. Let us compute the

[7]Our definition of the integral $\int_D f \, \mathrm{d}A$ is based on the *Riemannian* definition of integration. In Part II P&M you will be introduced to a different definition based on the *Lebesgue* notion of integrability. In the former you split up the *domain* of your function and add together bits from each part, whereas in the latter approach you split up the *range* of your function and add together bits from each part.

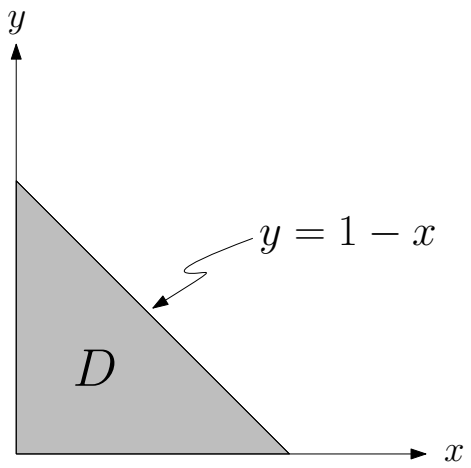

FIGURE 9. The region D.

integral of $f(x, y) = xy^2$ over D. Integrating over horizontal strips first

$$\begin{aligned}\int_D f \,\mathrm{d}A &= \int_0^1 \left(\int_0^{1-y} xy^2 \,\mathrm{d}x \right) \mathrm{d}y \\ &= \int_0^1 y^2 \left[\frac{1}{2} x^2 \right]_0^{1-y} \mathrm{d}y \\ &= \int_0^1 \frac{1}{2} y^2 (1-y)^2 \,\mathrm{d}y \\ &= \frac{1}{60}.\end{aligned}$$

If we instead integrate over vertical strips first

$$\begin{aligned}\int_D f \,\mathrm{d}A &= \int_0^1 \left(\int_0^{1-x} xy^2 \,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 x \left[\frac{1}{3} y^3 \right]_0^{1-x} \mathrm{d}x \\ &= \int_0^1 \frac{1}{3} x (1-x)^3 \,\mathrm{d}x \\ &= \frac{1}{60}.\end{aligned}$$

Note that if $f(x, y) = g(x)h(y)$ and D is a rectangle $\{(x, y) : a \le x \le b, c \le y \le d\}$ then things are much simpler

$$\int_D f \,\mathrm{d}A = \left(\int_a^b g(x) \,\mathrm{d}x \right) \left(\int_c^d h(y) \,\mathrm{d}y \right).$$

It is often convenient to introduce a change of variables in an integral to make the computation easier. For instance, making the change of variables $x = x(u)$ in a one-dimensional integral is

$$\int_a^b f(x) \,\mathrm{d}x = \int_\alpha^\beta f(x(u)) \frac{\mathrm{d}x}{\mathrm{d}u} \,\mathrm{d}u,$$

where $x(\alpha) = a$ and $x(\beta) = b$. There is a similar rule for changing variables in higher dimensional integrals.

Proposition. *Let $x = x(u,v)$ and $y = y(u,v)$ represent a smooth bijection with smooth inverse that maps the region D' in the (u,v)-plane to the region D in the (x,y)-plane. We write $\mathbf{x} = \mathbf{x}(u,v)$. Then*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\,\mathrm{d}v$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} = \det\left(\frac{\partial \mathbf{x}}{\partial u}\,\middle|\,\frac{\partial \mathbf{x}}{\partial v}\right)$$

is called the Jacobian, often denoted J. Short version: $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$.

In other words

$$\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v.$$

Note that we take the *modulus* of J. If the transformation between (x,y) and (u,v) is truly smooth and with smooth inverse, then J must be nonvanishing. In practice, it doesn't matter if J vanishes at a finite collection of points because they won't affect the value of the integral – they form a set of "measure zero", i.e. zero area.

We won't give a rigorous proof of this result, but the following argument gives the essential idea. A more detailed argument is found in Handout 2. We can compute the integral

$$\int_D f\,\mathrm{d}A$$

using sets A_{ij} that are formed between lines of the form $u(x,y) = u_i$ and $v(x,y) = v_j$ in the $(x,y)-$plane, where $u_{i+1} = u_i + \delta u$ and $v_{j+1} = v_j + \delta v$. Then δA_{ij} is approximately the area of the parallelogram generated by the vectors

$$\mathbf{x}(u_i+\delta u, v_j) - \mathbf{x}(u_i,v_j) \approx \frac{\partial \mathbf{x}}{\partial u}(u_i,v_j)\delta u, \quad \mathbf{x}(u_i, v_j+\delta v) - \mathbf{x}(u_i,v_j) \approx \frac{\partial \mathbf{x}}{\partial v}(u_i,v_j)\delta v.$$

Recalling that the area of a parallelogram generated by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$ we find

$$\delta A_{ij} \approx \left|\frac{\partial \mathbf{x}}{\partial u}(u_i,v_j)\delta u \times \frac{\partial \mathbf{x}}{\partial v}(u_i,v_j)\delta v\right| = |J(u_i,v_j)|\,\delta u\,\delta v.$$

Hence

$$\int_D f\,\mathrm{d}A = \lim_{\epsilon\to 0}\sum_{i,j} f(x(u_i,v_j),y(u_i,v_j))\,\delta A_{ij} = \iint_{D'} f(x(u,v),y(u,v))|J|\,\mathrm{d}u\,\mathrm{d}v.$$

We already know the left hand side is $\int_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, hence the result.

Example. Consider the transformation to polar coordinates (ρ,ϕ) defined by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

In this case

$$|J| = \left|\det\begin{pmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi}\end{pmatrix}\right| = \left|\det\begin{pmatrix}\cos\phi & -\rho\sin\phi\\ \sin\phi & \rho\cos\phi\end{pmatrix}\right| = |\rho| = \rho.$$

Let D be the region in the (x,y)-plane defined by

$$D = \{(x,y) : x>0,\ y>0,\ x^2+y^2<R^2\}.$$

This region is mapped in a one-to-one fashion the region D' in the (ρ, ϕ)-plane defined by

$$D' = \{(\rho, \phi) : 0 < \rho < R,\ 0 < \phi < \tfrac{\pi}{2}\}.$$

Hence

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(\rho\cos\phi, \rho\sin\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi.$$

Taking the limit $R \to \infty$, we conclude

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{\infty} f(\rho\cos\phi, \rho\sin\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi.$$

Consider the integral:

$$I = \int_0^{\infty} e^{-x^2}\,\mathrm{d}x.$$

Then using the previous result we find

$$\begin{aligned} I^2 &= \left(\int_0^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_0^{\infty} e^{-y^2}\,\mathrm{d}y\right) \\ &= \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{\infty} e^{-\rho^2}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &= \frac{\pi}{2}\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(-\frac{1}{2}e^{-\rho^2}\right)\mathrm{d}\rho \\ &= \frac{\pi}{4}. \end{aligned}$$

Taking the (positive) square root, we conclude that

$$\int_0^{\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

To integrate over regions V in $\mathbf{R}^3$ we use an entirely analogous approach

$$\int_V f(\mathbf{x})\,\mathrm{d}V = \lim_{\epsilon\to 0}\sum_{i,j,k} f(x_i, y_j, z_k)\,\delta V_{ijk}$$

where V_{ijk} are small sets, each contained inside a ball of radius ϵ, that partition the volume V. In this case the *volume element* satisfies

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and the three resulting integrals can be done in any order you like, as before.

Example. Consider the region V bound by the plane $x + y + z = 1$ and the three planes $x = 0, y = 0, z = 0$ as shown in figure 10.

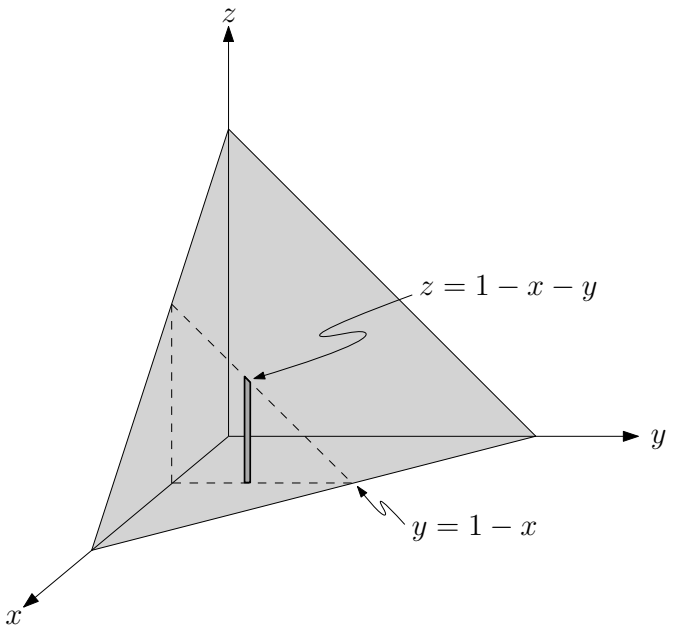

FIGURE 10. The volume V.

It has volume

$$\begin{aligned}\int_V \mathrm{d}V &= \int_{x=0}^{1} \mathrm{d}x \int_{y=0}^{1-x} \mathrm{d}y \int_{z=0}^{1-x-y} \mathrm{d}z \\ &= \int_{x=0}^{1} \mathrm{d}x \int_{y=0}^{1-x} (1-x-y)\,\mathrm{d}y \\ &= \int_{x=0}^{1} \frac{1}{2}(1-x)^2\,\mathrm{d}x \\ &= \frac{1}{6}\end{aligned}$$

Suppose V has constant density $\rho = 1$. Define[8] the center of mass of an object as

$$\mathbf{X} = \frac{1}{M}\int_V \rho\mathbf{x}\,\mathrm{d}V.$$

In this case, symmetry dictates that $\mathbf{X}$ is (r, r, r) for some $r > 0$.

$$r = \frac{1}{V}\int_V x\,\mathrm{d}V = 6\int_{x=0}^{1} \frac{1}{2}x(1-x)^2\,\mathrm{d}x = \frac{1}{4},$$

hence $\mathbf{X} = \frac{1}{4}(1,1,1)$.

Arguing in entirely the same way as we did for integrals in $\mathbf{R}^2$, we have the following.

Proposition. *Let $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ represent a smooth bijectino with smooth inverse that maps the volume V' in the (u,v,w)-region to the volume V in the (x,y,z)-region. We write $\mathbf{x} = \mathbf{x}(u,v,w)$. Then*

$$\iiint_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{V'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

[8]Since this definition implies $\int_V \rho(\mathbf{X}-\mathbf{x})\,\mathrm{d}V = M\mathbf{X} - \int_V \rho\mathbf{x}\,\mathrm{d}V = 0$.

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \det\left(\frac{\partial \mathbf{x}}{\partial u}\,\middle|\,\frac{\partial \mathbf{x}}{\partial v}\,\middle|\,\frac{\partial \mathbf{x}}{\partial w}\right)$$

is called the Jacobian, still often denoted J. Short version: $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$.

In this case the Jacobian comes from the fact that the parallelepiped generated by the vectors

$$\frac{\partial \mathbf{x}}{\partial u}\,\delta u, \quad \frac{\partial \mathbf{x}}{\partial v}\,\delta v, \quad \frac{\partial \mathbf{x}}{\partial w}\,\delta w$$

is equal to

$$\left|\frac{\partial \mathbf{x}}{\partial u}\,\delta u \cdot \left(\frac{\partial \mathbf{x}}{\partial v}\,\delta v \times \frac{\partial \mathbf{x}}{\partial w}\,\delta w\right)\right| = |J|\,\delta u\,\delta v\,\delta w.$$

Again we write $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$.

Example. If you go through the relevant algebra you find that in cylindrical coordinates

$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z,$$

and in spherical coordinates

$$\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Example. Suppose we want to calculate the volume of a sphere of radius R. In Cartesian coordinates this corresponds to the region

$$V = \{(x,y,z) : 0 \le x^2 + y^2 + z^2 \le R^2\}.$$

So the volume would be

$$\begin{aligned}
\int_V \mathrm{d}V &= \int_{z=-R}^{R} \mathrm{d}z \int_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \mathrm{d}y \int_{x=-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}} \mathrm{d}x \\
&= \int_{z=-R}^{R} \left(\int_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} 2\sqrt{R^2-z^2-y^2}\,\mathrm{d}y \right) \mathrm{d}z \\
&= \int_{z=-R}^{R} \left(y\sqrt{R^2-z^2-y^2} + (R^2-x^2)\arcsin\left[\frac{y}{\sqrt{R^2-x^2}}\right] \right) \Bigg|_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \mathrm{d}z \\
&= \int_{-R}^{R} \pi(R^2-z^2)\,\mathrm{d}z \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

Alternatively, if we use spherical polars:

$$V = \{(r,\theta,\phi) : 0 \le r \le R,\ 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi\},$$

and using $\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ we find

$$\int_V \mathrm{d}V = \int_{\phi=0}^{2\pi} \mathrm{d}\phi \int_{\theta=0}^{\pi} \mathrm{d}\theta \int_{r=0}^{R} \mathrm{d}r\, r^2 \sin\theta = \int_0^{\pi} \frac{2\pi R^3}{3} \sin\theta\,\mathrm{d}\theta = \frac{4\pi R^3}{3},$$

which was *much* easier.

Example. Consider a sphere of radius a with a cylinder of radius $b < a$ removed. The resulting volume is

$$V = \{(x, y, z) : 0 \leq x^2 + y^2 + z^2 \leq a^2,\ x^2 + y^2 \geq b^2\}.$$

Or, if we use cylindrical polar coordinates

$$V = \{(\rho, \phi, z) : b \leq \rho \leq a,\ 0 \leq z^2 + \rho^2 \leq a^2,\ 0 \leq \phi < 2\pi\}.$$

So the volume is

$$\int_V \mathrm{d}V = \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \int_{z=-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} \rho \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}z = 2\pi \int_b^a 2\rho\sqrt{a^2 - \rho^2}\,\mathrm{d}\rho = \frac{4}{3}\pi(a^2 - b^2)^{3/2}.$$

Don't try this using Cartesian coordinates!

3.4. **Integration over surfaces.** A two dimensional surface in $\mathbf{R}^3$ can be defined implicitly using a function $f : \mathbf{R}^3 \to \mathbf{R}$ via

$$S = \{\mathbf{x} : f(\mathbf{x}) = 0\}.$$

We already know that normal to the surface at $\mathbf{x}$ is parallel to $\nabla f(\mathbf{x})$. We call such a surface *regular* if $\nabla f \neq 0$ for $\mathbf{x} \in S$.

Example. If $f(\mathbf{x}) = x^2 + y^2 + z^2 - 1$ in Cartesian coordinates, then

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2\mathbf{x}$$

which is clearly normal to the sphere.

Some surfaces have a boundary, e.g. the hemisphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geq 0\}.$$

We label the boundary by ∂S, so that in this case

$$\partial S = \{(x, y, z) : x^2 + y^2 = 1,\ z = 0\}.$$

In this course, a surface S will either have no boundary ($\partial S = \emptyset$), or it have a boundary made of of a collection of piecewise smooth curves. Surfaces without a boundary are called *closed*, e.g. the sphere is a closed surface.

It is often useful to *parametrise* a surface using some coordinates (u, v), do that

$$S = \{\mathbf{x} = \mathbf{x}(u, v),\ (u, v) \in D\}$$

for some region D in the (u, v)-plane.

Example. Taking the hemisphere example as before, we can use spherical polar coordinates to write

$$S = \left\{ \mathbf{x}(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq \phi < 2\pi \right\}.$$

We call the parametrised surface $\mathbf{x} = \mathbf{x}(u, v)$ *regular* if

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0,$$

in which case we define the normal to be

$$\mathbf{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right|}.$$

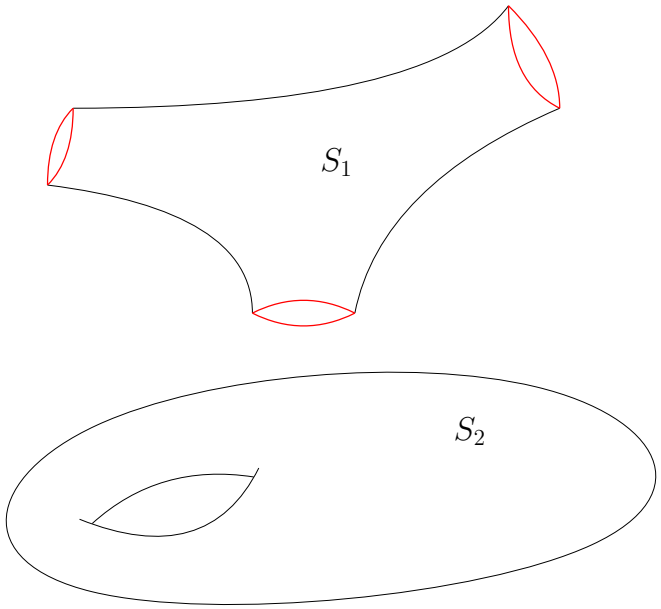

FIGURE 11. The boundary of S_1 is coloured in red, where as S_2 is a closed surface, similar to a torus.

Of course, we could always choose the the negative version of this. Once we've fixed the sign, we get a normal vector everywhere on our surface that varies smoothly from point to point. This also gives us a way to consistently orientate our the boundary of S: we fixed the convention that normal vectors in your immediate vicinity should be *on your left* as you traverse (each part of) the boundary of S. See figure 12.

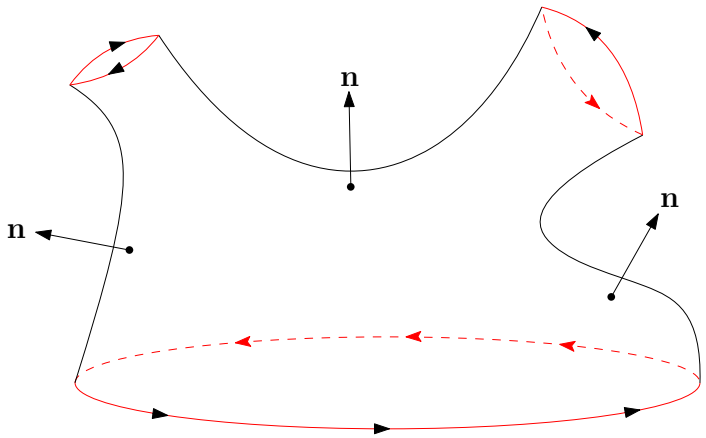

FIGURE 12. The orientation of ∂S is fixed by choosing a normal consistently over S.

How might we compute the area of a surface $S = \{\mathbf{x} = \mathbf{x}(u, v) : (u, v) \in D\}$? We might naively think it would be

$$\int_D \mathrm{d}u\,\mathrm{d}v$$

but this would be *wrong*, because a small change $(u, v) \mapsto (u + \delta u, v + \delta v)$ does not necessarily produce an area of $\delta u\, \delta v$ on S. A small change $u \mapsto u + \delta u$ produces a change

$$\mathbf{x}(u + \delta u, v) - \mathbf{x}(u, v) \approx \frac{\partial \mathbf{x}}{\partial u}\, \delta u.$$

Similarly, a small change $v \mapsto v + \delta v$ produces a change

$$\mathbf{x}(u, v + \delta v) - \mathbf{x}(u, v) \approx \frac{\partial \mathbf{x}}{\partial v}\, \delta v.$$

So to leading order, the change in $(u, v) \mapsto (u + \delta u, v + \delta v)$ produces a parallelogram generated by $\frac{\partial \mathbf{x}}{\partial u}\delta u$ and $\frac{\partial \mathbf{x}}{\partial v}\delta v$ which has area

$$\left| \frac{\partial \mathbf{x}}{\partial u}\, \delta u \times \frac{\partial \mathbf{x}}{\partial v}\, \delta v \right| = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \delta u\, \delta v.$$

This leads us to define the *scalar area* element and *vector area* element by

$$\mathrm{d}S = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v, \quad \mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\, \mathrm{d}u\, \mathrm{d}v = \mathbf{n}\, \mathrm{d}S.$$

So the surface area of S is

$$\mathrm{area}(S) = \int_S \mathrm{d}S = \iint_D \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v.$$

Example. Consider the hemisphere with radius R

$$S = \left\{ \mathbf{x}(\theta, \phi) = \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix} \equiv R\mathbf{e}_r,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi < 2\pi \right\}.$$

We see that

$$\frac{\partial \mathbf{x}}{\partial \theta} = \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = R\mathbf{e}_\theta, \quad \frac{\partial \mathbf{x}}{\partial \phi} = \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix} = R\sin\theta\, \mathbf{e}_\phi.$$

Hence

$$\mathrm{d}S = R^2 \sin\theta |\mathbf{e}_\theta \times \mathbf{e}_\phi|\, \mathrm{d}\theta\, \mathrm{d}\phi = R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi.$$

Consequently

$$\mathrm{area}(S) = \int_{\theta=0}^{\pi/2} \mathrm{d}\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi\, R^2 \sin\theta = 2\pi R^2$$

as expected.

Suppose we model a fluid flow with the vector field $\mathbf{u} = \mathbf{u}(\mathbf{x})$. Given a surface S, how do we calculate how much fluid passes through S per unit time? On a small patch δS on S, the volume of fluid crossing it in time δt is $(\mathbf{u} \cdot \delta\mathbf{S})\delta t$. So total amount of fluid that passes over S in time δt is

$$\delta t \int_S \mathbf{u} \cdot \mathrm{d}\mathbf{S}.$$

So the integral gives the rate at which fluid is passing through S. These type are often called *flux* integrals, they arise all over the place in physics.

It is worth checking that our definition of the surface area is independent of the particular parametrisation we use to describe S. Let $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \tilde{\mathbf{x}}(\tilde{u}, \tilde{v})$ be two different parametrisations of the same surface S with $(u, v) \in D$ and $(\tilde{u}, \tilde{v}) \in D'$. Then we must have a relationship:

$$\mathbf{x}(u, v) = \tilde{\mathbf{x}}(\tilde{u}(u, v), \tilde{v}(u, v))$$

for appropriate functions $\tilde{u}(u,v)$ and $\tilde{v}(u,v)$. By the chain rule

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial u}\times\frac{\partial \mathbf{x}}{\partial v} &= \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u}+\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u}\right)\times\left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial v}+\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial v}\right)\\ &= \left(\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v}-\frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u}\right)\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\times\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\\ &= \frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\times\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\right)\end{aligned}$$

Hence

$$\iint_{D'}\left|\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\times\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\right|\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}=\iint_{D}\left|\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\times\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\right|\left|\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v=\iint_{D}\left|\frac{\partial \mathbf{x}}{\partial u}\times\frac{\partial \mathbf{x}}{\partial v}\right|\mathrm{d}u\,\mathrm{d}v$$

i.e. the definition of $\int_S \mathrm{d}S$ is independent of the choice of parametrisation.

4. Divergence, Curl and the Laplacian

4.1. **Definitions.** We have already seen the gradient operator ∇ which acts on scalar functions $f : \mathbf{R}^3 \to \mathbf{R}$. In Cartesian coordinates

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}.$$

We can use ∇ to define new differential operators that act on vector fields.

For a vector field $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ we define the *divergence* of $\mathbf{F}$ by $\nabla \cdot \mathbf{F}$. In Cartesian coordinates

$$\nabla \cdot \mathbf{F} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot (F_j \mathbf{e}_j) = (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial F_j}{\partial x_i},$$

and using $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ we find that in Cartesian coordinates

$$\boxed{\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}}$$

Note that in our computation the derivatives just pass through the $\{\mathbf{e}_j\}$ because these basis vectors are independent of (x, y, z). Of course, this is not the case with a generic curvilinear coordinate system! The divergence of a vector field is a scalar function.

We define the *curl* of $\mathbf{F}$ by $\nabla \times \mathbf{F}$. In Cartesian coordinates

$$\nabla \times \mathbf{F} = \left(\mathbf{e}_j \frac{\partial}{\partial x_j} \right) \times (F_k \mathbf{e}_k) = (\mathbf{e}_j \times \mathbf{e}_k) \frac{\partial F_k}{\partial x_j} = \mathbf{e}_i \left(\epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \right),$$

where we used $\mathbf{e}_j \times \mathbf{e}_k = \epsilon_{ijk} \mathbf{e}_i$. So in Cartesian coordinates

$$\boxed{[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}}$$

We can also express this as a formal determinant

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} \equiv \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

Note that the definition of the divergence holds in any number of dimensions, but the definition of curl is special to three dimensions because it involves the vector product.

Finally, for a scalar field $f : \mathbf{R}^3 \to \mathbf{R}$ we define the Laplacian of f, written $\nabla^2 f$, via the formula

$$\nabla^2 f = \nabla \cdot \nabla f$$

so that in Cartesian coordinates

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}}$$

which follows from the definition of the divergence and $F_i = \partial f / \partial x_i$.

Example. Consider the vector field $\mathbf{F}(\mathbf{x}) = \mathbf{x}$. Then using suffix notation and Cartesian coordinates

$$\nabla \cdot \mathbf{F} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3$$

and

$$[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial x_k}{\partial x_j} = \epsilon_{ijk} \delta_{jk} = \epsilon_{ijj} = 0.$$

Proposition. *For scalar functions f, g and vector fields $\mathbf{F}, \mathbf{G}$ we have*

$$\begin{aligned}
\nabla(fg) &= (\nabla f)g + (\nabla g)f \\
\nabla \cdot (f\mathbf{F}) &= (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F}) \\
\nabla \times (f\mathbf{F}) &= (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F}) \\
\nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} \\
\nabla \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \\
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}).
\end{aligned}$$

Proof. We use suffix notation and Cartesian coordinates. We will just do the fifth one, as they are all very similar. Be sure to note that, for example,

$$[(\mathbf{F} \cdot \nabla)\mathbf{G}]_i = \left(F_j \frac{\partial}{\partial x_j}\right) G_i = F_j \frac{\partial G_i}{\partial x_j}.$$

The left hand side is

$$\begin{aligned}
[\nabla \times (\mathbf{F} \times \mathbf{G})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\mathbf{F} \times \mathbf{G})_k \\
&= \epsilon_{ijk}\epsilon_{klm} \frac{\partial}{\partial x_j} (F_l G_m) \\
&= (\delta_{li}\delta_{jm} - \delta_{lj}\delta_{im}) \left(F_l \frac{\partial G_m}{\partial x_j} + G_m \frac{\partial F_l}{\partial x_j}\right) \\
&= F_i \frac{\partial G_j}{\partial x_j} - G_i \frac{\partial F_j}{\partial x_j} + G_j \frac{\partial F_i}{\partial x_j} - F_j \frac{\partial G_i}{\partial x_j} \\
&= [\mathbf{F}(\nabla \cdot \mathbf{G})]_i - [\mathbf{G}(\nabla \cdot \mathbf{F})]_i + [(\mathbf{G} \cdot \nabla)\mathbf{F}]_i - [(\mathbf{F} \cdot \nabla)\mathbf{G}]_i
\end{aligned}$$

and the result follows. The others are similar. ∎

All these identities hold in any coordinate system, but they are most easily established using Cartesian coordinates.

Since our definitions of the divergence and curl are coordinate independent, we can use them for an arbitrary set of orthogonal curvilinear coordinates, For example[9]

$$\nabla \cdot \mathbf{F} = \left(\mathbf{e}_u \frac{1}{h_u}\frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v}\frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w}\frac{\partial}{\partial w}\right) \cdot (F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w)\,.$$

However, care must be taken because the basis vectors $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ will, in general, depend on the coordinates (u, v, w), so we cannot simply pass the derivatives through them as we did with Cartesian coordinates. It turns out that this is quite cumbersome – the best way is to use some equivalent definitions of the divergence and curl in terms of integrals, but that will have to wait until the next section. We simply state the relevant results:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_w h_u F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right]$$

[9]Note the ordering: the differential operator acts first. So you interpret such terms as, for example

$$\left(\mathbf{e}_u \frac{\partial}{\partial u}\right) \cdot (F_v \mathbf{e}_v) = \mathbf{e}_u \cdot \left(\frac{\partial}{\partial u}(F_v \mathbf{e}_v)\right) = \mathbf{e}_u \cdot \left(\frac{\partial F_v}{\partial u}\,\mathbf{e}_v + F_v \frac{\partial \mathbf{e}_v}{\partial u}\right) = F_v \left(\mathbf{e}_u \cdot \frac{\partial \mathbf{e}_v}{\partial u}\right).$$

and

$$\nabla \times \mathbf{F} = \frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v)\right]\mathbf{e}_u + \text{cyclic perms.}$$

Since we know that $(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u}$ etc. in orthogonal curvilinear coordinates, we get from our expression for $\nabla \cdot \mathbf{F}$ that

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right].$$

For example, in cylindrical polars

$$\begin{aligned}\nabla^2 f &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right] \\ &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.\end{aligned}$$

Performing a similar calculation in spherical polars gives

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 f}{\partial \phi^2}.$$

Calculating the Laplacian of a vector field is a little subtle. You might simply write

$$\nabla \cdot (\nabla \mathbf{F})$$

but we have not defined $\nabla\mathbf{F}$. If we do things in Cartesian coordinates, then because the basis vectors are independent of the coordinates it makes sense to define

$$\nabla^2 \mathbf{F} = (\nabla^2 F_i)\,\mathbf{e}_i.$$

You can show, using suffix notation that for Cartesian coordinates this gives

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

The right hand side of this expression is well defined in *any* coordinate system, so we use it as the definition of the Laplacian on vector fields.

There are some important relationships between each of the differential operators we've seen so far.

Proposition. *For a scalar field f and vector field $\mathbf{F}$ we have*

$$\nabla \times \nabla f = 0, \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

i.e. curl $\circ$ grad $= 0$ and div $\circ$ curl $= 0$.

Proof. Using Cartesian coordinates

$$[\nabla \times \nabla f]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = \epsilon_{ijk}\frac{\partial^2 f}{\partial x_j \partial x_k} = 0$$

owning to the symmetry of the partial derivatives. Similarly

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x_i}\left(\epsilon_{ijk}\frac{\partial F_k}{\partial x_j}\right) = \epsilon_{ijk}\frac{\partial^2 F_k}{\partial x_i \partial x_j} = 0.$$

■

Recall $\mathbf{F}$ was *conservative* if $\mathbf{F} = \nabla f$. We have just seen that if $\mathbf{F}$ is conservative then *necessarily* $\nabla \times \mathbf{F} = 0$. We say $\mathbf{F}$ is *irrotational* if its curl vanishes. Hence

$$\mathbf{F} \text{ conservative } \Rightarrow \mathbf{F} \text{ is irrotational.}$$

The reverse implication is true if the domain Ω on which $\mathbf{F}$ is defined is simply connected, i.e. all closed curves can be continuously shrunk to a point. We might call this property *1-connected*. Note that $\Omega = \mathbf{R}^3$ is 1-connected, but $\Omega = \mathbf{R}^3 \setminus \{x = 0, y = 0\}$ is not, since we've removed the z-axis, so a loop that enclosed the z-axis cannot be continuously shrunk to a point in Ω.

Similarly, we've seen that if there exists a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$ then $\nabla \cdot \mathbf{F} = 0$. We call $\mathbf{A}$ a *vector potential* for $\mathbf{F}$. We say $\mathbf{F}$ is *solenoidal* if its divergence vanishes. So we have the implication

$$\text{existence of vector potential for } \mathbf{F} \Rightarrow \mathbf{F} \text{ is solenoidal.}$$

The reverse implication holds if the domain Ω on which $\mathbf{F}$ is defined is 1-connected and has the additional topological property: every sphere in Ω can be continuously shrunk to any point in Ω. We call domains with this property[10] *2-connected*. For example, $\Omega = \mathbf{R}^3$ is 2-connected, but $\Omega = \{\mathbf{x} : 0 < |\mathbf{x}| \leq 2\}$ is not, because a sphere containing the origin cannot be continuously shrunk to an arbitrary point in Ω.

4.2. **Topology via calculus (non-examinable).** For $\Omega \subset \mathbf{R}^3$ consider a vector field $\mathbf{F} : \Omega \to \mathbf{R}^3$. If $\mathbf{F} = \nabla f$ for some scalar function f, then we've shown $\nabla \times \mathbf{F} = 0$, i.e.

$$\mathbf{F} \text{ conservative } \Rightarrow \mathbf{F} \text{ is irrotational.}$$

This result doesn't depend on Ω. The reverse implication is more interesting. We know that *if Ω is simply connected* (or equivalently, *1-connected*), i.e. all closed loops in Ω can be continuously deformed to any point in Ω without leaving it, then the reverse implication is true.

$(\star)$ $\quad \mathbf{F} : \Omega \to \mathbf{R}^3$ irrotational and Ω simply connected $\Rightarrow$ $\mathbf{F}$ is conservative.

Using this result we can get results about topology without doing any topology.

Example. Is the domain

$$\Omega = \mathbf{R}^3 \setminus \{z\text{-axis}\}$$

simply connected or not? We could wave our arms around, talking about closed loops, but a pure mathematician wouldn't accept these arm-waves as proofs. However, they do know that the statement $(\star)$ is true. Let's use this to prove that Ω is not simply connected.

Consider the vector field $\mathbf{F} : \Omega \to \mathbf{R}^3$ defined by

$$\mathbf{F}(\mathbf{x}) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

We see that $\mathbf{F}$ really is well defined on all of Ω. Also

$$\nabla \times \mathbf{F}(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

[10] Our definitions of 1- and 2-connectivity are a little wishy-washy. The proper definitions involve concepts from algebraic topology and in particular the notion of a *homotopy group*. These clever groups contain important information about the topology of a domain. We say Ω is n-connected if its first n homotopy groups, $\pi_1(\Omega), \ldots, \pi_n(\Omega)$ are trivial. You can learn about these in Part II algebraic topology.

i.e. $\mathbf{F}$ is irrotational on all of Ω.

Let us suppose that Ω is simply connected. Then $(\star)$ implies that $\mathbf{F} = \nabla f$ for some f. So for any closed loop C in Ω we have

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \oint_C \nabla f \cdot \mathrm{d}\mathbf{x} = 0.$$

Consider the curve $C = \{z = 0, x^2 + y^2 = 1\}$. We can parametrise this curve via

$$[0, 2\pi] \ni t \mapsto \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}.$$

So

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi \neq 0.$$

This is a contradiction, so we conclude that Ω is *not* simply connected.

That's how you do topology without doing any topology.

5. Integral theorems

5.1. **Green's theorem: statement and examples.** Green's theorem is the simplest of the integral theorems we will see.

Proposition. *If $P = P(x, y)$ and $Q = Q(x, y)$ are continuously differentiable functions on A and ∂A is made from a collection of piecewise smooth curves, then*

$$\oint_{\partial A} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

The orientation of the boundary ∂A is such that A lies to your left as you traverse it.

The orientation of the boundary is consistent with our previous definition if you imagine the planar region having a normal that sticks out of the page. So going around the boundary with the normal on your left is the same as going along the boundary with the region A on your left. See the following.

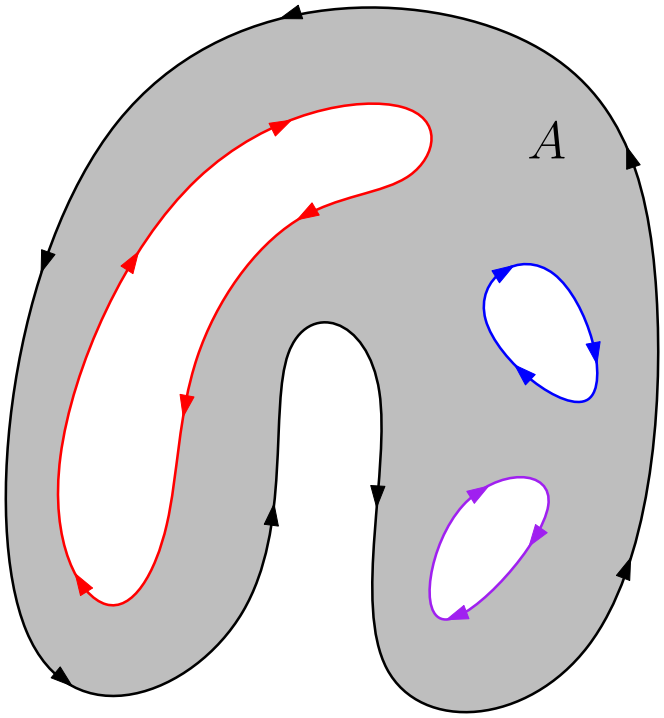

Figure 13. Orientation of the boundary of the planar region A.

We will give an idea of how to prove this result and others towards the end of this section. Until then, it is worth noting that the result certainly holds on a rectangle

$$A = \{(x, y) : a \le x \le b,\ c \le y \le d\}$$

since in this case the right hand side of Green's theorem is

$$\int_c^d \mathrm{d}y \int_a^b \frac{\partial Q}{\partial x}\,\mathrm{d}x - \int_a^b \mathrm{d}x \int_c^d \frac{\partial P}{\partial y}\,\mathrm{d}y$$

and using the fundamental theorem of calculus this becomes

$$\int_c^d (Q(b, y) - Q(a, y))\,\mathrm{d}y + \int_a^b (P(x, c) - P(x, d))\,\mathrm{d}x = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Example. Let $(P, Q) = (-\frac{1}{2}y, \frac{1}{2}x)$. Then Green's theorem tells us

$$\text{area}(A) = \iint_A \mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\oint_{\partial A} x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Let A be the ellipse $x^2/a^2 + y^2/b^2 = 1$ so that C has parametrisation

$$[0, 2\pi] \ni t \mapsto \left(\begin{smallmatrix} a\cos t \\ b\sin t \end{smallmatrix}\right).$$

Then

$$\text{area}(A) = \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,\mathrm{d}t = \pi ab,$$

as expected.

5.2. **Stokes' theorem: statement and examples.** Stokes' theorem, like Green's theorem, relates a one dimensional integral to a two dimensional integral. But here the two dimensional integral can be a surface in $\mathbf{R}^3$.

Proposition. *If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field and S is an orientable, piecewise regular surface with piecewise regular boundary ∂S then*

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x}.$$

The orientable bit is important: it means that there's a consistent choice of unit normal $\mathbf{n}$ on S that varies continuously from point to point. You can think of orientable surfaces as those which have two "sides". For example, a cylinder is orientable whereas the Möbius strip is *not* orientable – it only has one side.

Example. Let S be a section of the sphere defined in spherical coordinates by

$$S = \left\{ \mathbf{x}(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \equiv \mathbf{e}_r : 0 \le \theta \le \alpha,\ 0 \le \phi < 2\pi \right\}.$$

Let $\mathbf{F} = (-x^2 y, 0, 0)$ so that $\nabla \times \mathbf{F} = (0, 0, x^2)$. Using

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi}\,\mathrm{d}\theta\,\mathrm{d}\phi = \mathbf{e}_\theta \times (\sin\theta\,\mathbf{e}_\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi = \mathbf{e}_r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and the fact that $x^2(\mathbf{e}_z \cdot \mathbf{e}_r) = (\sin\theta\cos\phi)^2\cos\theta$ on S we find the surface integral to be

$$\int_{\phi=0}^{2\pi} \mathrm{d}\phi\,\cos^2\phi \int_{\theta=0}^{\alpha} \mathrm{d}\theta\,\sin^3\theta\cos\theta = \pi \times \frac{1}{4}\sin^4\alpha = \frac{\pi}{4}\sin^4\alpha.$$

The boundary is described by the curve

$$[0, 2\pi] \ni t \mapsto \begin{pmatrix} \sin\alpha\cos t \\ \sin\alpha\sin t \\ \cos\alpha \end{pmatrix}, \quad \text{hence} \quad \mathrm{d}\mathbf{x} = \sin\alpha \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \mathrm{d}t.$$

And so we get

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \sin^4\alpha \int_0^{2\pi} (-\cos^2 t \sin t)(-\sin t)\,\mathrm{d}t = \frac{\pi}{4}\sin^4\alpha.$$

Example. Let S be an orientable, closed surface. Then for any continuously differentiable vector field $\mathbf{F}$

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = 0.$$

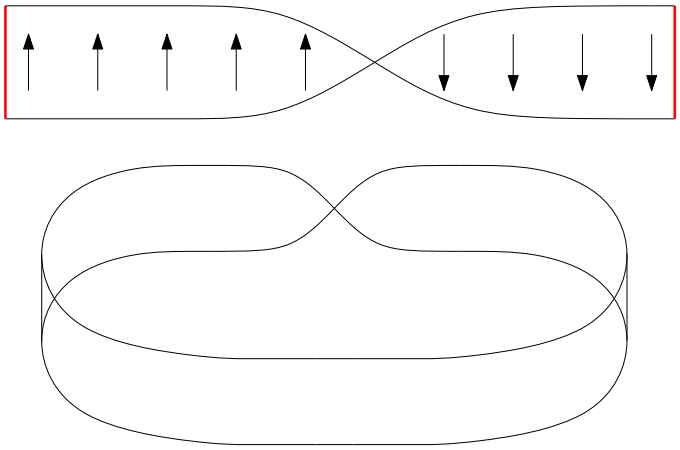

FIGURE 14. A Möbius strip formed by identifying the two red lines.

Proposition. *If* $\mathbf{F}$ *is continuously differentiable and for every simple loop* C

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

then $\nabla \times \mathbf{F} = 0$.

Proof. Suppose the result is false, i.e. $\mathbf{k} \cdot (\nabla \times \mathbf{F}) > 0$ at $\mathbf{x} = \mathbf{x}_0$ for some unit vector $\mathbf{k}$. So there exists an $\epsilon > 0$ such that

$$\mathbf{k} \cdot (\nabla \times \mathbf{F})(\mathbf{x}_0) = \epsilon > 0.$$

By the continuity[11] of $\nabla \times \mathbf{F}$, for $\delta > 0$ sufficiently small we have

$$\mathbf{k} \cdot (\nabla \times \mathbf{F})(\mathbf{x}) > \tfrac{1}{2}\epsilon \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_0| < \delta.$$

Taking a simple loop inside this ball which lies in a plane with normal $\mathbf{k}$ then gives

$$0 = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{k} \cdot (\nabla \times \mathbf{F}) \, \mathrm{d}S > \tfrac{1}{2}\epsilon \int_S \mathrm{d}S$$

which is a contradiction. We conclude that a vector field is irrotational if it has zero circulation about any closed loop. ■

The converse of the above is true if the domain of the vector field is simply connected.

Example. Let S_ϵ denote a region contained inside a disc of radius $\epsilon > 0$ centred at the point $\mathbf{x}_0$ with normal $\mathbf{k}$. So

$$\int_{S_\epsilon} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \mathrm{area}(S_\epsilon)\, \mathbf{k} \cdot (\nabla \times \mathbf{F})(\mathbf{x}_0) + \int_{S_\epsilon} (\nabla \times \mathbf{F}(\mathbf{x}) - \nabla \times \mathbf{F}(\mathbf{x}_0)) \cdot \mathrm{d}\mathbf{S}.$$

[11] Recall that a function $f : \mathbf{R}^3 \to \mathbf{R}$ is continuous at the point $\mathbf{x}_0$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{x}_0| < \delta$ then $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$. So if $f(\mathbf{x}_0) = \epsilon$ then $f(\mathbf{x}) > \frac{1}{2}\epsilon$ if $|\mathbf{x} - \mathbf{x}_0| < \delta$.

By continuity of $\nabla \times \mathbf{F}$, the second term tends to zero faster[12] than area(S_ϵ) so by taking the limit and using Stokes' theorem we find

$$\mathbf{k} \cdot (\nabla \times \mathbf{F})(\mathbf{x}_0) = \lim_{\epsilon \to 0} \frac{1}{\text{area}(S_\epsilon)} \oint_{\partial S_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{x}.$$

This gives another coordinate independent definition of curl. It tells us that the component of $\nabla \times \mathbf{F}$ pointing along in the $\mathbf{k}$ axis is equal to the infinitesimal circulation around that axis per unit area.

5.3. **Möbius strips and Stokes (non-examinable).** The orientable nature of S is *important*! Consider the Möbius strip with parametrisation

$$S = \left\{ \mathbf{x}(u,v) = \begin{pmatrix} (1 + \frac{v}{2}\cos\frac{u}{2})\cos u \\ (1 + \frac{v}{2}\cos\frac{u}{2})\sin u \\ \frac{v}{2}\sin\frac{u}{2} \end{pmatrix},\ 0 \le u < 2\pi,\ -1 \le v \le 1 \right\}.$$

This surface is *not* orientable. We saw in the last section that the vector field

$$\mathbf{F}(\mathbf{x}) = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

is irrotational, i.e. $\nabla \times \mathbf{F} = 0$. If we tried to use Stokes theorem on S with this choice of $\mathbf{F}$ we would conclude

$$0 = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x}.$$

However, the boundary of the Möbius strip is

$$[0, 4\pi] \ni t \mapsto \begin{pmatrix} (1 + \frac{1}{2}\cos\frac{t}{2})\cos t \\ (1 + \frac{1}{2}\cos\frac{t}{2})\sin t \\ \frac{1}{2}\sin\frac{t}{2} \end{pmatrix}.$$

Note that t has to travel all the way upto 4π: if you start at the "twist", you see that the $0 \le t < 2\pi$ takes you along the top of the band, whereas $2\pi \le t < 4\pi$ takes you along the bottom, as you can see from Figure 14. So the line integral would be

$$\int_0^{4\pi} \frac{1}{(1 + \frac{1}{2}\cos\frac{t}{2})} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{4}\sin\frac{t}{2}\cos t - (1 + \frac{1}{2}\cos t)\sin t \\ -\frac{1}{4}\sin\frac{t}{2}\sin t + (1 + \frac{1}{2}\cos\frac{t}{2})\cos t \\ \frac{1}{4}\cos\frac{t}{2} \end{pmatrix} \mathrm{d}t = \int_0^{4\pi} \mathrm{d}t = 4\pi.$$

This shows us that Stokes' theorem is *not* applicable if the surface is not orientable.

5.4. **Divergence theorem: statement and examples.** The divergence theorem (or Gauss' theorem) relates volume integrals to surface integrals, or area integrals to line integrals. The three dimensional version is as follows.

Proposition. *If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ be a continuously differentiable vector field and V is a volume with a piecewise regular boundary ∂V then*

$$\int_V \nabla \cdot \mathbf{F}\, \mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where the normal to ∂V points outwards from V.

[12] We have the estimate

$$\frac{1}{\text{area}(S_\epsilon)} \left| \int_{S_\epsilon} (\nabla \times \mathbf{F}(\mathbf{x}) - \nabla \times \mathbf{F}(\mathbf{x}_0)) \cdot \mathrm{d}\mathbf{S} \right| \le \sup_{|\mathbf{x}-\mathbf{x}_0| \le \epsilon} |\nabla \times \mathbf{F}(\mathbf{x}) - \nabla \times \mathbf{F}(\mathbf{x}_0)|$$

which tends to zero by the continuity of $\nabla \times \mathbf{F}$ at $\mathbf{x}_0$.

And the two dimensional version is.

Proposition. *Let $\mathbf{F} = \mathbf{F}(\mathbf{x})$ be a continuously differentiable vector field and $D \subset \mathbf{R}^2$ a region with piecewise smooth boundary ∂D then*

$$\int_D \nabla \cdot \mathbf{F} \, \mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s$$

where the normal to ∂D points outwards from D.

Example. Let V be a cylinder, defined using cylindrical polars (ρ, ϕ, z) by

$$V = \{(\rho, \phi, z) : 0 \leq \rho \leq R, \ -h \leq z \leq h, \ 0 \leq \phi < 2\pi\}.$$

Consider the vector field $\mathbf{F} = \mathbf{x}$. Then

$$\int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V = 3 \int_V \mathrm{d}V = 6\pi h R^2.$$

Alternatively we can use the divergence theorem to compute the same thing via a surface integral. The boundary ∂V is made of of three surfaces

$$S_R = \{(\rho, \phi, z) : \rho = R, \ -h \leq z \leq h, \ 0 \leq \phi < 2\pi\}$$

and

$$S_\pm = \{(\rho, \phi, z) : 0 \leq \rho \leq R, \ z = \pm h, \ 0 \leq \phi < 2\pi\}.$$

On S_R we have $\mathrm{d}\mathbf{S} = \mathbf{e}_\rho \, R \, \mathrm{d}\phi \, \mathrm{d}z$ and $\mathbf{x} \cdot \mathbf{e}_\rho = R$ so

$$\int_{S_R} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{z=-h}^{h} \int_{\phi=0}^{2\pi} R^2 \, \mathrm{d}\phi \, \mathrm{d}z = 4\pi R^2 h.$$

Whereas on $S_\pm$ we have $\mathrm{d}\mathbf{S} = \pm \mathbf{e}_z \, \rho \, \mathrm{d}\rho \, \mathrm{d}\phi$ and $\mathbf{e}_z \cdot \mathbf{x} = \pm h$, so

$$\int_{S_\pm} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{R} h\rho \, \mathrm{d}\rho \, \mathrm{d}\phi = \pi R^2 h.$$

In summary

$$\int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \int_{S_+} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \int_{S_-} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 6\pi R^2 h,$$

which agrees with the volume integral.

Proposition. *If $\mathbf{F}$ is continuously differentiable and for every closed surface S*

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0$$

then $\nabla \cdot \mathbf{F} = 0$.

Proof. Suppose the result is false and $\nabla \cdot \mathbf{F} > 0$ at $\mathbf{x} = \mathbf{x}_0$, i.e.

$$\nabla \cdot \mathbf{F}(\mathbf{x}_0) = \epsilon > 0$$

for some $\epsilon > 0$. By continuity, we must have

$$\nabla \cdot \mathbf{F}(\mathbf{x}) > \tfrac{1}{2}\epsilon \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_0| < \delta$$

for δ sufficiently small. Choosing a volume V inside this ball gives

$$0 = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V > \tfrac{1}{2}\epsilon \int_V \mathrm{d}V,$$

which is a contradiction. We conclude that a vector field is solenoidal if it has zero net flux across any closed surface. ∎

The converse of this result is true if the domain of the vector field is 2-connected.

Example. Let V_ϵ denote a volume in $\mathbf{R}^3$ contained inside a ball of radius $\epsilon > 0$, centred at the point $\mathbf{x}_0$. Then

$$\int_{V_\epsilon} \nabla \cdot \mathbf{F} \, \mathrm{d}V = \mathrm{vol}(V_\epsilon) \nabla \cdot \mathbf{F}(\mathbf{x}_0) + \int_{V_\epsilon} (\nabla \cdot \mathbf{F}(\mathbf{x}) - \nabla \cdot \mathbf{F}(\mathbf{x}_0)) \, \mathrm{d}V.$$

By the continuity of $\nabla \cdot \mathbf{F}$, the second term tends to zero faster[13] than $\mathrm{vol}(V_\epsilon)$ so by taking the limit and using the divergence theorem we find

$$\nabla \cdot \mathbf{F}(\mathbf{x}_0) = \lim_{\epsilon \to 0} \frac{1}{\mathrm{vol}(V_\epsilon)} \int_{\partial V_\epsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

This gives another coordinate independent definition of the divergence. It tells us that $\nabla \cdot \mathbf{F}$ measures the infinitesimal flux per unit volume. So $\nabla \cdot \mathbf{F}(\mathbf{x}_0) > 0$ implies there is a kind of *source* at $\mathbf{x}_0$, more things flow away from that point than towards it. Similarly $\nabla \cdot \mathbf{F}(\mathbf{x}_0) < 0$ indicates a sink. If $\nabla \cdot \mathbf{F} = 0$ then the field $\mathbf{F}$ can be thought of as representing the velocity of some *incompressible* fluid, meaning that there is no net build up of fluid in any given region.

Example. Consider vector field $\mathbf{F}(\mathbf{x}) = \mathbf{x}$. Then taking $\mathbf{x}_0 = 0$ and $V_\epsilon = \{\mathbf{x} : |\mathbf{x}| < \epsilon\}$ in the previous example we get

$$\nabla \cdot \mathbf{F}(0) = \lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\epsilon \mathbf{e}_r \cdot \mathbf{e}_r) \, \epsilon^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = 3.$$

Alternatively, using Cartesian coordinates

$$\nabla \cdot \mathbf{F} = \frac{\partial x_i}{\partial x_i} = 3.$$

Example. Many equations in physics can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

for some scalar function $\rho = \rho(\mathbf{x}, t)$ and vector field $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$. Such equations are called *conservation laws*. If both ρ and $\mathbf{J}$ decay sufficiently rapidly as $|\mathbf{x}| \to \infty$, we define the *conserved charge* by

$$Q = \int_{\mathbf{R}^3} \rho(\mathbf{x}, t) \, \mathrm{d}V.$$

This quantity really is conserved, since

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{\mathbf{R}^3} \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\int_{\mathbf{R}^3} \nabla \cdot \mathbf{J} \, \mathrm{d}V = -\lim_{R \to \infty} \int_{|\mathbf{x}|=R} \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0,$$

assuming $\mathbf{J} \cdot \mathbf{e}_r$ decays sufficiently rapidly as $|\mathbf{x}| \to \infty$. Hence

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0,$$

so Q really is a conserved quantity.

[13] We have the estimate

$$\frac{1}{\mathrm{vol}(V_\epsilon)} \left| \int_{V_\epsilon} (\nabla \cdot \mathbf{F}(\mathbf{x}) - \nabla \cdot \mathbf{F}(\mathbf{x}_0)) \, \mathrm{d}V \right| \leq \sup_{|\mathbf{x}-\mathbf{x}_0| \leq \epsilon} |\nabla \cdot \mathbf{F}(\mathbf{x}) - \nabla \cdot \mathbf{F}(\mathbf{x}_0)|$$

which tends to zero by continuity of $\nabla \cdot \mathbf{F}$ at $\mathbf{x}_0$.

5.5. **Noether's theorem (non-examinable).** There is a beautiful theorem of mathematical physics, called Noether's theorem, that says that the existence of a conservation law is equivalent to the existence of a symmetry in your underlying set of equations. Emmy Noether proved this theorem in 1918. It pulls together results from the theory of Lie groups, Lie algebras, infinite prolongations of group actions, Lagrangian mechanics and gives an amazing insight into why things are the way they are. Many physics and applied maths books give a loose version of this result, but the full version, which is essentially what Noether proved herself back in 1918, can be found in Peter Olver's excellent book *Applications of Lie Groups to Differential Equations.* You might want to look it up after you've finished your 3rd year.

So onto the ramifications of Noether's result. Have you ever asked yourself: where do the laws of conservation of energy, momentum, angular momentum etc. actually come from? Why does nature want to preserve them? Why does the Universe react in such a way as to keep these things from changing? Noether's theorem tells you why!

The laws of nature don't (usually), depend on what time of day it is, where you've placed your lab or which direction you've pointed in. That is to say, many of the universes laws have time-symmetry, translational symmetry and rotational symmetry. Noether tells us that, to each of these symmetries, there is a conservation law. We've ended up calling these conserved things "energy", "linear momentum" and "angular momentum", but they're all just bi-products of the symmetries that exist in our laws of nature.

Energy is conserved because mother nature doesn't wear a watch.

Emmy Noether taught us this.

5.6. **Sketch proofs.** Here we give a basic idea of how to prove the divergence theorem, from which we can get Green's theorem and Stokes' theorem for free – it turns out these all these integral theorems are the same result dressed up in a different way[14].

Proposition. *The Divergence Theorem is true.*

Proof. One way of proving this is to observe that

Gauss said it is true, therefore, it is true.

If that isn't convincing enough for you, read on. Initially suppose that $\mathbf{F}$ only has a non-zero component in the $\mathbf{e}_z$ direction, so that

$$\mathbf{F}(\mathbf{x}) = F_z(x, y, z)\,\mathbf{e}_z.$$

The divergence theorem then says

$$\int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V = \int_{\partial V} F_z \mathbf{e}_z \cdot \mathrm{d}\mathbf{S}.$$

Let us compute both sides of this statement and show that they are indeed equal.

[14]All these integral theorems can be show to be equivalent to what's confusingly called *Stokes' theorem* as well. It is a result in differential geometry that says that if you have a (higher-dimensional-analogue-of) surface M with boundary ∂M, then

$$\int_M \mathrm{d}\eta = \int_{\partial M} \eta.$$

Here η is (higher-dimensional-analogue-of) a differential form and $\mathrm{d}\eta$ and *exterior derivative* of η. It should remind you of the fundamental theorem of calculus. Stokes' theorem is a corollary of Stokes' theorem.

Consider the projection of ∂V onto the (x, y)-plane. We assume V is simple so that this map is just 2-to-1, i.e. the surface ∂V has a simple upper and lower side.

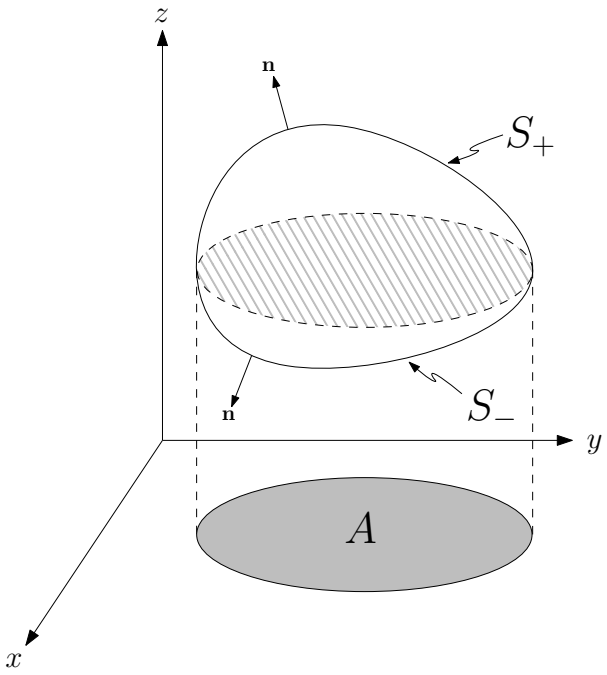

FIGURE 15. Projection of convex volume V onto (x, y)-plane.

We can describe the upper and lower parts of S via

$$S_\pm = \left\{ \mathbf{x}(x,y) = \begin{pmatrix} x \\ y \\ g_\pm(x,y) \end{pmatrix},\ (x,y) \in A \right\}.$$

The volume integral becomes

$$(\star) \qquad \int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V = \iint_A \left[\int_{g_-(x,y)}^{g_+(x,y)} \frac{\partial F_z}{\partial z}\,\mathrm{d}z \right] \mathrm{d}x\,\mathrm{d}y$$
$$= \iint_A \Big[F_z(x,y,g_+(x,y)) - F_z(x,y,g_-(x,y)) \Big] \mathrm{d}x\,\mathrm{d}y.$$

To calculate the surface integrals over $S_\pm$ we use

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \begin{pmatrix} -\partial g_\pm/\partial x \\ -\partial g_\pm/\partial y \\ 1 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y.$$

We want the normal to point *out* of V, and in particular we want the z-component of the normal to be negative on S_-, as seen in figure 15. So we adjust the sign of $\mathrm{d}\mathbf{S}$ on S_- so that on $S_\pm$ the vector area elements are

$$\mathrm{d}\mathbf{S} = \pm \begin{pmatrix} -\partial g_\pm/\partial x \\ -\partial g_\pm/\partial y \\ 1 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y.$$

Hence

$$\begin{aligned}\int_S F_z\mathbf{e}_z\cdot\mathrm{d}\mathbf{S} &= \left[\int_{S_+}+\int_{S_-}\right]F_z\mathbf{e}_z\cdot\mathrm{d}\mathbf{S}\\ &= \iint_A F_z(x,y,g_+(x,y))\mathrm{d}x\,\mathrm{d}y-\iint_A F_z(x,y,g_-(x,y))\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_V\frac{\partial F_z}{\partial z}\,\mathrm{d}V,\end{aligned}$$

where the final line follows from $(\star)$. In exactly the same we we can show that

$$\int_V\frac{\partial F_x}{\partial x}\,\mathrm{d}V=\int_S F_x\mathbf{e}_x\cdot\mathrm{d}\mathbf{S},\quad \int_V\frac{\partial F_y}{\partial y}\,\mathrm{d}V=\int_S F_y\mathbf{e}_y\cdot\mathrm{d}\mathbf{S}.$$

By summing up each of these results we conclude

$$\int_V\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)\mathrm{d}V=\int_S(F_x\mathbf{e}_x+F_y\mathbf{e}_y+F_z\mathbf{e}_z)\cdot\mathrm{d}\mathbf{S},$$

which is the general result. The same method of proof holds for the two dimensional version of the divergence theorem. Our proof only holds for simple regions (e.g. convex), but can be modified so that it works on much more complex regions. ■

Given we know the divergence theorem is true, in any number of dimensions, we can immediately get proofs of Green's theorem and Stokes' theorem. We will show

$$\text{Divergence theorem}\ \Rightarrow\ \text{Green's theorem}\ \Rightarrow\ \text{Stokes' theorem}.$$

(a) **Divergence theorem ⇒ Green's theorem**

First off, using the two dimensional divergence theorem with $\mathbf{F}=(Q,-P)$ we have

$$\iint_A\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\int_A\nabla\cdot\mathbf{F}\,\mathrm{d}A=\oint_{\partial A}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s.$$

If ∂A is parametrised with respect to arc-length, then the unit tangent vector is given by $\mathbf{t}=(x'(s),y'(s))$. This implies that the unit normal vector, which points *out* of A, must be given by $\mathbf{n}=(y'(s),-x'(s))$. Easy to check: if $\mathbf{t}$ points vertically upwards then A is on left, meaning $\mathbf{n}$ should be pointing to the right, which it does. So we have

$$\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s=Q\frac{\mathrm{d}y}{\mathrm{d}s}\,\mathrm{d}s+P\frac{\mathrm{d}x}{\mathrm{d}s}\,\mathrm{d}s=P\,\mathrm{d}x+Q\,\mathrm{d}y$$

and we recover Green's theorem.

(b) **Green's theorem ⇒ Stoke's theorem**

Consider a regular parametrised surface $S=\{\mathbf{x}=\mathbf{x}(u,v):(u,v)\in A\}$ with corresponding boundary $\partial S=\{\mathbf{x}=\mathbf{x}(u,v):(u,v)\in\partial A\}$. Green's theorem tells us

$$\int_{\partial A}P\,\mathrm{d}u+Q\,\mathrm{d}v=\iint_A\left(\frac{\partial Q}{\partial u}-\frac{\partial P}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v.$$

We apply this result to the specific choices

$$P(u,v)=\mathbf{F}(\mathbf{x}(u,v))\cdot\frac{\partial\mathbf{x}}{\partial u},\quad Q(u,v)=\mathbf{F}(\mathbf{x}(u,v))\cdot\frac{\partial\mathbf{x}}{\partial v}.$$

First we note that

$$P\,\mathrm{d}u+Q\,\mathrm{d}v=\mathbf{F}(\mathbf{x}(u,v))\cdot\left(\frac{\partial\mathbf{x}}{\partial u}\,\mathrm{d}u+\frac{\partial\mathbf{x}}{\partial v}\,\mathrm{d}v\right)=\mathbf{F}(\mathbf{x}(u,v))\cdot\mathrm{d}\mathbf{x}(u,v)$$

which gives the left hand side as

$$\int_{\partial A} P\,\mathrm{d}u + Q\,\mathrm{d}v = \int_{\partial S} \mathbf{F}\cdot\mathrm{d}\mathbf{x}.$$

For the other side we compute the necessary derivatives using the chain rule

$$\frac{\partial Q}{\partial u} = \frac{\partial x_j}{\partial u}\frac{\partial F_i}{\partial x_j}\frac{\partial x_i}{\partial v} + F_i\frac{\partial^2 x_i}{\partial u\partial v}$$

and similarly

$$\frac{\partial P}{\partial v} = \frac{\partial x_j}{\partial v}\frac{\partial F_i}{\partial x_j}\frac{\partial x_i}{\partial u} + F_i\frac{\partial^2 x_i}{\partial u\partial v}.$$

This gives

$$\begin{aligned}\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} &= \left(\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial u} - \frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}\right)\frac{\partial F_i}{\partial x_j}\\ &= \left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right)\frac{\partial F_i}{\partial x_j}\frac{\partial x_p}{\partial v}\frac{\partial x_q}{\partial u}\\ &= \epsilon_{ijk}\epsilon_{pqk}\frac{\partial F_i}{\partial x_j}\frac{\partial x_p}{\partial v}\frac{\partial x_q}{\partial u}\\ &= (\nabla\times\mathbf{F})\cdot\left(\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}\right)\end{aligned}$$

meaning that

$$\iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v = \iint_A (\nabla\times\mathbf{F})\cdot\left(\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v = \int_S (\nabla\times\mathbf{F})\cdot\mathrm{d}\mathbf{S}.$$

So we have Stokes' theorem from Green's theorem. Note that this proof also works if the boundary of A is a collection of disjoint curves.

6. Maxwell's Equations

6.1. **Brief introduction to electromagnetism.** Maxwell's wonderful equations are a triumph of 19th century physics. That a set of four relatively simple equations can describe all that we know about classical electromagnetic fields is utterly amazing.

The two vector fields relevant to electromagnetism are the *magnetic field*, denoted by $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$; and the *electric field*, denoted by $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$. These fields interact. They depend on each other, as well as the *charge density* $\rho = \rho(\mathbf{x}, t)$ (electric charge per unit volume), and the *current density* $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ (electric current per unit area). Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_o \mathbf{J}. \tag{4}$$

If you know the charge density ρ and the current density $\mathbf{J}$, you should be able to use these equations to figure out how the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ will evolve. The constants ε_0 and μ_0 refer to the permittivity and permeability of free space, which obey the relation

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

where $c = 299,792,458\,\mathrm{ms}^{-1}$ is the speed of light.

Taking the divergence of the fourth and using $\nabla \cdot \nabla \times \mathbf{B} = 0$ gives

$$0 = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) + \mu_0 \nabla \cdot \mathbf{J}.$$

The first equation implies that the first term is $\mu_0 \frac{\partial \rho}{\partial t}$, so we get the conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This gives rise to *conservation of charge*[15]

6.2. **Integral formulation.** Maxwell's equations can be put in an equivalent form involving integrals by using the integral theorems we've seen earlier in the course. Integrating Maxwell's first equation over a volume V and using the divergence theorem gives

$$\int_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V \equiv \frac{Q}{\varepsilon_0},$$

where Q denotes the total charge inside the volume V. This is known as Gauss' law for electric fields. The equivalent statement for magnetic fields is

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0.$$

[15]As Noether told us: this conservation law correspond to a symmetry of the underlying equations. In this case the symmetry is associated with what's called the total *gauge invariance* of the electromagnetic field. You can learn about this in Part II Electrodynamics or in §6.5.

As expected, this means there is not net magnetic flux over the closed surface ∂V. That is to say, there are no magnetic monopoles[16]. This highlights a fundamental difference between the electric and magnetic fields.

Integrating the third of Maxwell's equations over a surface S and using Stokes' theorem yields

$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

So the rate of change of magnetic flux across the surface S produces a circulation of the electric field $\mathbf{E}$ about ∂S. In other words, a changing magnetic field will induce a current.

Finally, integrating the fourth of Maxwell's equations over a surface S and using Stokes' gives

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}.$$

We see that electric current produces circulation of the magnetic field about the axis of direction of the current. So if current flows along a wire, a magnetic field will be produced that circulated the wire in the positive sense. You might have seen this in school, where you predict the direction of the magnetic field using the right hand rule.

6.3. **Electromagnetic waves.** In empty space we have $\rho = 0$ and $\mathbf{J} = 0$, so Maxwell's equations become

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= 0. \end{aligned}$$

Recall that the Laplacian of a vector field is defined by

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}).$$

So from Maxwell's equations (1), (3) and (4) we get

$$\nabla^2 \mathbf{E} = -\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

If we set $\mu_0 \epsilon_0 = 1/c^2$, we get the wave equation for the electric field

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

This tells us that, in a vacuum, the electric field travels at the speed of light. Similarly, using Maxwell's equations (2), (3) and (4) we find

$$\nabla^2 \mathbf{B} = -\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

So again we get the wave equation, but this time for the magnetic field

$$\boxed{\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0}$$

[16]Or are there...?

6.4. **Electrostatics and Magnetostatics.** When all the fields and source terms are independent of time, Maxwell's equations simplify. The equations decouple into

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \mathbf{E} = 0$$

and

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

If our domain is all of $\mathbf{R}^3$, which is 2-connected, then $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ imply

$$\mathbf{E} = -\nabla\phi \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Here ϕ is called the *electric potential* for $\mathbf{E}$ and $\mathbf{A}$ is called the *magnetic potential* for $\mathbf{B}$. Maxwell's equations are then

$$-\nabla^2\phi = \frac{\rho}{\varepsilon_0}, \quad \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}.$$

We will discuss how to solve the first of these in §7.

6.5. **Gauge invariance (non-examinable).** Let us return to the case of the magnetic potential. Since we always have from (2) that

$$\nabla \cdot \mathbf{B} = 0$$

we can always write $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$, assuming we work on all of $\mathbf{R}^3$. Important observation: this definition does not define $\mathbf{A}$ uniquely, because we can add $\nabla\chi$ to $\mathbf{A}$, where χ is a scalar function. Doing so does not change the value of $\mathbf{B} = \nabla \times \mathbf{A}$ since $\nabla \times \nabla\chi = 0$. This is called *gauge invariance*. We have a certain amount of freedom in how we define $\mathbf{A}$. This invariance encapsulates the precise symmetry that generates conservation of charge, via Noether's theorem.

Using $\mathbf{B} = \nabla \times \mathbf{A}$ in (3) we see

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

This implies the existence of a scalar potential ϕ such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}.$$

So we can reduce Maxwell's equations down to

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{\rho}{\varepsilon_0},$$

and

$$\nabla \times (\nabla \times \mathbf{A}) + \mu_0\varepsilon_0 \nabla \left(\frac{\partial \phi}{\partial t} \right) + \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{J}.$$

Recalling that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ and $\mu_0\varepsilon_0 = 1/c^2$ we find that the second of these equations can be rewritten in the form

$$-\left(\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) + \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = \mu_0 \mathbf{J}.$$

Now we can exploit gauge freedom: without loss of generality we can assume that

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0.$$

Indeed, by changing $\mathbf{A}$ to $\mathbf{A} + \nabla\chi$ we can make sure this term vanishes, using a suitable function χ. Maxwell's equations are then reduced to

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}.$$

These are called Maxwell's equations in *Lorenz gauge*. The electric and magnetic fields can be recovered from the formulas

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

7. Poisson's equation

7.1. **The boundary value problem.** Many equations in mathematical physics can be reduced to one of the form

$$\nabla^2\varphi = F.$$

We call this *Poisson's equation*. This equation either needs to be solved on all of $\mathbf{R}^n$, or some domain $\Omega \in \mathbf{R}^n$. We will be interested in the cases $n = 2, 3$. When $F = 0$ we call the resulting equation *Laplace's equation*, which we discuss in more detail in §7.5.

Physical problems involve boundary conditions, so to solve Poisson's equation we need to prescribe some boundary conditions for the function φ on the boundary $\partial\Omega$, or as $|\mathbf{x}| \to \infty$ if solving on all of $\mathbf{R}^n$. The ***Dirichlet problem*** for Poisson's equation is

$$\begin{cases} \nabla^2\varphi = F, & \text{in } \Omega, \\ \varphi = f, & \text{on } \partial\Omega, \end{cases}$$

and the ***Neumann problem*** is

$$\begin{cases} \nabla^2\varphi = F, & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\mathbf{n}} = g, & \text{on } \partial\Omega. \end{cases}$$

Here we have used the normal derivative

$$\frac{\partial\varphi}{\partial\mathbf{n}} \equiv \mathbf{n}\cdot\nabla\varphi.$$

It is important to interpret the boundary conditions in an appropriate way: we assume that the solution *continuously* approaches the boundary data as $\mathbf{x}$ approaches the boundary $\partial\Omega$ from within Ω, i.e. we assume that φ and $\nabla\varphi$ are continuous on the set $\Omega \cup \partial\Omega$.

Note that for a φ to satisfy $\nabla^2\varphi = 0$ in the domain Ω, it is necessary that φ be well-defined on all of Ω. So don't fall into the trap of assuming things like

$$\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = 0$$

for all $\mathbf{x} \in \mathbf{R}^3$. Whilst it is true for $\mathbf{x} \neq 0$ (check this!), it is certainly *not* true at $\mathbf{x} = 0$, since the function isn't well-defined there. So when we're talking about solutions to Laplace's equation, we're necessarily talking about functions which are non-singular.

Example. As usual let $r = |\mathbf{x}|$. Consider the boundary value problem

$$\begin{cases} \nabla^2\varphi = r, & \text{in } r < a, \\ \varphi = 1, & \text{on } r = a. \end{cases}$$

The spherical symmetry suggests we look for a solution of the form $\varphi = \varphi(r)$. Substituting in and using the fact for spherically symmetric φ

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)$$

we see that our problem is equivalent to

$$\left(r^2\varphi'\right)' = r^3, \quad \varphi(a) = 1.$$

The complementary function must satisfy $r^2\varphi' = \text{const}$, which has general solution

$$\varphi(r) = A + \frac{B}{r},$$

for some constants A, B. However the second term is singular at $r = 0$, which is part of our domain, so we must have $B = 0$. The particular integral is seen to be $\varphi(r) = \frac{1}{12}r^3$. So the general solution is

$$\varphi(r) = A + \frac{1}{12}r^3.$$

The boundary condition at $r = a$ gives

$$\varphi(r) = 1 + \frac{1}{12}\left(r^3 - a^3\right).$$

We only want to solve problems that have a unique, or almost unique solution. Let us consider a generic linear problem of the form

$$\begin{cases} L\varphi = F, & \text{in } \Omega, \\ B\varphi = f, & \text{on } \partial\Omega, \end{cases}$$

for some linear differential operators L and B. If φ_1 and φ_2 are two different solutions to this problem, then their difference, $\psi = \varphi_2 - \varphi_1$, must satisfy the homogeneous problem

$$\begin{cases} L\psi = 0, & \text{in } \Omega, \\ B\psi = 0, & \text{on } \partial\Omega. \end{cases}$$

If we can show that the only solution to this problem is $\psi = 0$, we will have to conclude that $\varphi_1 = \varphi_2$, i.e. the solution to the original problem is unique. The moral of the story

The solution to a linear problem is unique iff the only solution to homogeneous problem is the zero solution.

Proposition. *The solution to the Dirichlet problem is unique. The solution to the Neumann problem is unique upto the addition of a constant.*

Proof. Let φ be the solution to the homogeneous

$$\begin{cases} \nabla^2\varphi = 0, & \text{in } \Omega, \\ B\varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $B\varphi \equiv \varphi$ (Dirichlet) or $B\varphi \equiv \frac{\partial\varphi}{\partial\mathbf{n}}$ (Neumann). Consider the non-negative functional

$$I[\varphi] = \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}V \geq 0.$$

Clearly $I[\varphi] = 0$ if and only if $\nabla\varphi = 0$ on Ω. Note that

$$I[\varphi] = \int_\Omega \nabla\varphi \cdot \nabla\varphi \,\mathrm{d}V = \int_\Omega \Big(\nabla \cdot (\varphi\nabla\varphi) - \varphi\nabla^2\varphi\Big)\mathrm{d}V.$$

The second term vanishes because $\nabla^2\varphi = 0$ in Ω. Using the divergence theorem on the first term, we find

$$I[\varphi] = \int_{\partial\Omega} \varphi\frac{\partial\varphi}{\partial\mathbf{n}} \,\mathrm{d}S = 0,$$

hence $\nabla\varphi = 0$ throughout Ω, i.e. φ is constant. We then have two cases to consider: (a) In the Dirichlet case, $\varphi = 0$ on the boundary. By continuity of φ on $\Omega \cup \partial\Omega$ we conclude $\varphi = 0$ throughout Ω. So the solution to the Dirichlet problem is unique. (b)

In the Neumann case we only know that $\frac{\partial\varphi}{\partial\mathbf{n}} = 0$ on $\partial\Omega$, so we can't say any more than $\varphi = \text{const}$ throughout Ω. So the solution to the Neumann problem is unique upto the addition of a constant. ■

So solving the Dirichlet problem gives you a unique solution, whilst the most general solution to the Neumann problem is of the form $\varphi + \text{const}$.

Example. Consider a charge density of the form

$$\rho(\mathbf{x}) = \begin{cases} 0, & r < a \\ F(r), & r \geq a. \end{cases}$$

We claim that there is no electric field in the region $r < a$. Indeed, in $r < a$ we know the potential satisfies

$$\nabla^2\phi = -\frac{\rho(\mathbf{x})}{\varepsilon_0} = 0.$$

By spherical symmetry we see that $\phi = \phi(r)$. In particular, ϕ is constant on the boundary $r = a$, i.e. $\phi(a) = \phi_0$. Now note that $\phi(\mathbf{x}) = \phi_0$ solves Laplace's equation inside the ball $r < a$ and has the correct boundary behaviour, so by uniqueness we have found the solution. Since ϕ is constant throughout $r < a$, we can conclude that $\mathbf{E}(\mathbf{x}) = 0$ there.

The same example holds if we replace electrostatics with gravity. We've essentially just proved Newton's shell theorem: there is no net gravitational field anywhere inside a spherical shell. Newton's proof was significantly more involved, but only used elementary concepts from geometry. It's also not very rigorous.

7.2. **Gauss' flux method.** A very clever way of getting solutions to Poisson's equation is to use the *Gauss method*. Suppose the source term F is spherically symmetric, i.e. $F = F(r)$ where $r = |\mathbf{x}|$. Re-write our problem as

$$\nabla \cdot \nabla\varphi = F(r).$$

Since the right hand side only depends on r, the same is true of the left hand side. So we can assume $\varphi = \varphi(r)$ also, and in particular $\nabla\varphi = \varphi'(r)\,\mathbf{e}_r$. Integrating our equation over a sphere of radius $R > 0$ and using the divergence theorem gives

$$\int_{|\mathbf{x}|=R} \nabla\varphi \cdot \mathrm{d}\mathbf{S} = \int_{|\mathbf{x}|<R} F(r)\,\mathrm{d}V \equiv Q(R),$$

where $Q(R)$ denotes the total amount of "stuff" with density $F(r)$ in $|\mathbf{x}| < R$. Since $\nabla\varphi \cdot \mathbf{e}_r = \varphi'(r)$ is a constant on $|\mathbf{x}| = R$, the integral on the left hand side can be done explicitly. Using $\mathrm{d}S = R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ we find

$$\varphi'(R) \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = Q(R).$$

Performing the θ and ϕ-integrals we deduce

$$\varphi'(R) = \frac{1}{R^2} \int_0^R r^2 F(r)\,\mathrm{d}r \equiv \frac{Q(R)}{4\pi R^2}$$

Assuming that $\varphi \to 0$ as $r \to \infty$, we can integrate up to find

$$\varphi(r) = -\int_r^\infty \left(\frac{1}{R^2} \int_0^R s^2 F(s)\,\mathrm{d}s \right) \mathrm{d}R.$$

Example. We find the electric field produced by the spherically symmetric charge density

$$\rho(r) = \begin{cases} \rho_0, & 0 \le r \le a, \\ 0, & r > a. \end{cases}$$

Recall that the electric potential ϕ defined by $\mathbf{E} = -\nabla\phi$. Use Maxwell's first equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{i.e.} \quad \nabla^2\phi = -\frac{\rho}{\varepsilon_0}.$$

By the previous formula we find

$$\phi'(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}.$$

Note that if $r > a$ then $Q(r) = Q(a) = Q$, where Q denotes the total charge. We get

$$\mathbf{E}(\mathbf{x}) = -\phi'(r)\mathbf{e}_r = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}\,\mathbf{e}_r, & r \le a \\ \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\mathbf{e}_r, & r > a. \end{cases}$$

Taking $a \to 0$ we deduce that the electric field to a point charge Q at the origin is

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{x}}{|\mathbf{x}|^3}.$$

In this case the relevant charge density is $\rho(\mathbf{x}) = Q\delta(\mathbf{x})$ and the corresponding electric potential is

$$\phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}|}.$$

Gauss' method also works if there's cylindrical symmetry. Suppose our problem takes the form

$$\nabla^2\varphi = F(\rho)$$

where $\rho^2 = x^2 + y^2$. This time if we integrate $\nabla \cdot \nabla\varphi = F$ over a cylinder of radius R, height a and using $\nabla\varphi = \varphi'(\rho)\,\mathbf{e}_\rho$ the divergence theorem gives

$$\varphi'(R)\int_{z=0}^{a}\int_{\phi=0}^{2\pi} R\,\mathrm{d}\phi\,\mathrm{d}z = \int_{\rho=0}^{R}\int_{z=0}^{a}\int_{\phi=0}^{2\pi} F(\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z,$$

Note that we don't get any contribution from the ends of the cylinder because $\mathbf{n} \cdot \mathbf{e}_\rho = 0$ on them. Performing the ϕ and z-integrals, we find

$$\varphi'(\rho) = \frac{1}{\rho}\int_0^\rho sF(s)\,\mathrm{d}s.$$

Example. How might we describe the charge density of a *line* of charge with charge density λ per unit length? Let $F(\rho)$ be the desired charge density. Then for any cylinder of length 1, we would want the integral of the charge density over that cylinder to be λ, i.e.

$$\lambda = \int_{z=a}^{a+1}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{b} F(\rho)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z = 2\pi\int_0^b \rho F(\rho)\,\mathrm{d}\rho.$$

So we take the charge density to be

$$F(\rho) = \frac{\lambda\delta(\rho)}{2\pi\rho}.$$

The electric potential would satisfy

$$\phi'(\rho) = -\frac{1}{\rho}\int_0^\rho \frac{\lambda}{2\pi}\delta(s)\,\mathrm{d}s = -\frac{\lambda}{2\pi\rho}.$$

The electric field is then

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{2\pi\rho}\,\mathbf{e}_\rho.$$

7.3. **Superposition principle.** Linear problems are comparatively easy because of one central fact: if

$$L\psi_n = F_n \quad n = 1, 2, 3, \ldots \quad \text{then} \quad L\left(\sum_n \psi_n\right) = \sum_n F_n.$$

So we can *superimpose* solutions. It is sometimes easiest to break up the forcing term F into several parts $F = \sum_n F_n$, then solve each of the problems separately

$$L\psi_n = F_n$$

Finally add them together to get the solution to the full problem $\psi = \sum_n \psi_n$.

Example. Consider the electric potential due to two point charges Q_a and Q_b located at the points $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$. The corresponding charge density is

$$\rho(\mathbf{x}) = Q_a\delta(\mathbf{x} - \mathbf{a}) + Q_b\delta(\mathbf{x} - \mathbf{b}).$$

We solve the problem via superposition. For the one point charge, the electric potential obeys

$$-\nabla^2\phi = \frac{Q_a}{\varepsilon_0}\delta(\mathbf{x} - \mathbf{a}).$$

By a shift of the origin we know the corresponding electric potential field is

$$\phi(\mathbf{x}) = \frac{Q_a}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x} - \mathbf{a}|}.$$

So, by superimposing the electric fields produced by each separate point charge, we deduce that the electric potential due to point charges at $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$ is

$$\phi(\mathbf{x}) = \frac{Q_a}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x} - \mathbf{a}|} + \frac{Q_b}{4\pi\varepsilon_0}\,\frac{1}{|\mathbf{x} - \mathbf{b}|}.$$

Example. Consider the electric potential outside a ball $|\mathbf{x}| < R$ of uniform charge density ρ_0 that has have several balls removed from its interior,

$$|\mathbf{x} - \mathbf{a}_i| < R_i, \quad i = 1, \ldots, N,$$

with $|\mathbf{a}_i| + R_i < R$ and $|\mathbf{a}_i - \mathbf{a}_j| > R_i + R_j$ for each $i \neq j$. See figure 16 below. To compute the potential in $|\mathbf{x}| > R$ we use the superposition principle. We consider each hole to be a ball of uniform charge density $-\rho_0$. Then the effective electric potential from each hole is

$$\phi_i(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\frac{Q_i}{|\mathbf{x} - \mathbf{a}_i|} \quad \text{on} \quad |\mathbf{x} - \mathbf{a}_i| > R_i,$$

where $Q_i = (4\pi R_i^3/3)\rho_0$. Using the potential for the ball on $|\mathbf{x}| > R$ with $Q = (4\pi R^3/3)\rho_0$ and the superposition principle, we deduce that the total electric potential for this Swiss cheese charge distribution on $|\mathbf{x}| > R$ is

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{|\mathbf{x}|} - \sum_{i=1}^{N}\frac{Q_i}{|\mathbf{x} - \mathbf{a}_i|}\right].$$

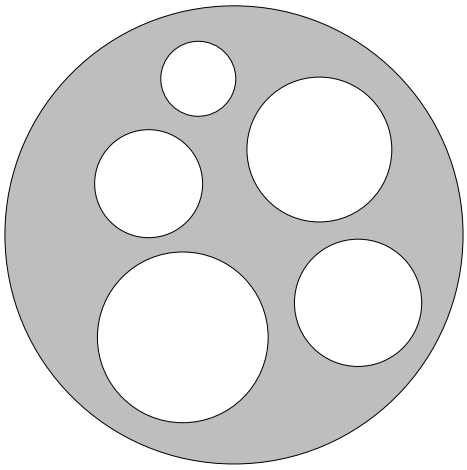

FIGURE 16. A swiss cheese charge density.

7.4. **Integral Solutions.** We have already seen, in our discussion in the previous section, that the electric potential due to a point charge at $\mathbf{x} = \mathbf{a}$ is proportional to

$$\frac{1}{|\mathbf{x} - \mathbf{a}|}.$$

We also know that a sum of point charges has, via the superposition principle, an electric potential consisting of a sum

$$\sum_i \frac{Q_i}{|\mathbf{x} - \mathbf{a}_i|}.$$

This leads us to consider very general superposition of potentials of the form

$$\int_{\mathbf{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}V(\mathbf{y}).$$

Proposition. *The unique solution to Dirichlet problem*

$$\begin{cases} \nabla^2 \varphi = F, & \text{in } \mathbf{R}^3, \\ |\varphi(\mathbf{x})| \to 0, & \text{as } |\mathbf{x}| \to \infty, \end{cases}$$

(assuming F has sufficient decay as $|\mathbf{x}| \to \infty$) is given by

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}V.$$

This result is equivalent to saying

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) = \delta(\mathbf{x}),$$

since by differentiating (with respect to the $\mathbf{x}$-variables) under the integral sign

$$\int_{\mathbf{R}^3} \nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) F(\mathbf{y}) \, \mathrm{d}V = \int_{\mathbf{R}^3} \delta(\mathbf{x} - \mathbf{y}) F(\mathbf{y}) \, \mathrm{d}V = F(\mathbf{x}).$$

You can prove this rigorously on Example Sheet 3, on one of the additional problems. Using suffix notation and Cartesian coordinates we have for $r \neq 0$

$$\frac{\partial^2}{\partial x_i \partial x_i} \left(\frac{1}{r} \right) = \frac{\partial}{\partial x_i} \left(-\frac{x_i}{r^3} \right) = -\frac{\delta_{ii}}{r^3} + \frac{3 x_i x_i}{r^5} = 0.$$

So certainly

$$\nabla^2\left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\right) = \delta(\mathbf{x}) \quad \text{for } \mathbf{x} \neq 0.$$

If we assume the divergence theorem works with delta functions, then for any ball $|\mathbf{x}| < R$

$$\begin{aligned}\int_{|\mathbf{x}|<R} \nabla^2\left(\frac{1}{|\mathbf{x}|}\right) \mathrm{d}V &= \int_{|\mathbf{x}|=R} \nabla\left(\frac{1}{r}\right) \cdot \mathrm{d}\mathbf{S} \\ &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left(-\frac{\mathbf{e}_r}{R^2}\right) \cdot \mathbf{e}_r \, R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= -4\pi\end{aligned}$$

So for any $R > 0$ we have

$$\int_{|\mathbf{x}|<R} \nabla^2\left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\right) \mathrm{d}V = 1 = \int_{|\mathbf{x}|<R} \delta(\mathbf{x}) \, \mathrm{d}V.$$

This is exactly the same behaviour as the delta function $\delta(\mathbf{x})$. So we conclude

$$\nabla^2\left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\right) = \delta(\mathbf{x}),$$

and the result in the proposition follows. This is an example of a *Green's function*[17]. You will see a lot more of these in the IB Methods course.

7.5. **Harmonic Functions.** When the forcing term F in Poisson's equation is zero, we call the resulting equation *Laplace's equation*

$$\nabla^2\varphi = 0.$$

Functions that satisfy Laplace's equation are said to be *harmonic*. Harmonic functions have lots of nice properties, like the following: the value of a harmonic function at a point $\mathbf{a}$ is equal to the average of the values it takes on any sphere centred at $\mathbf{a}$.

Proposition. *If φ is harmonic on $\Omega \subset \mathbf{R}^3$ then*

$$\varphi(\mathbf{a}) = \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \, \mathrm{d}S,$$

for $\mathbf{a} \in \Omega$ and r sufficiently small.

Proof. Let $F(r)$ denote the right hand side of our expression. Then

$$(\dagger) \qquad F(r) = \frac{1}{4\pi r^2}\int_{|\mathbf{x}|=r} \varphi(\mathbf{a}+\mathbf{x}) \, \mathrm{d}S = \frac{1}{4\pi}\int_0^{2\pi}\left[\int_0^{\pi} \varphi(\mathbf{a}+r\mathbf{e}_r)\sin\theta \, \mathrm{d}\theta\right] \mathrm{d}\phi.$$

[17]Same Green as Green's theorem – he was busy!

Differentiating under the integral sign and using $\frac{\mathrm{d}}{\mathrm{d}r}\varphi(\mathbf{a}+r\mathbf{e}_r) = \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)$.

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}r} &= \frac{1}{4\pi}\int_0^{2\pi}\left[\int_0^{\pi} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)\sin\theta\,\mathrm{d}\theta\right]\mathrm{d}\phi \\ &= \frac{1}{4\pi r^2}\int_{|\mathbf{x}|=r} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+\mathbf{x})\,\mathrm{d}S \\ &= \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r} \nabla\varphi \cdot \mathrm{d}\mathbf{S} \\ &= \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2\varphi\,\mathrm{d}V \\ &= 0.\end{aligned}$$

So F is constant and from (†) we see $\lim_{r\to 0} F(r) = \varphi(\mathbf{a})$, hence the result. ■

If we examine the proof of this result more closely, we can get an intuitive idea of what the Laplacian measures. We can also see that the converse to the previous theorem is true.

Proposition. *For any smooth scalar field φ on $\mathbf{R}^3$*

$$\nabla^2\varphi(\mathbf{a}) = \lim_{r\to 0}\frac{6}{r^2}\left[\frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r}\varphi(\mathbf{x})\,\mathrm{d}S - \varphi(\mathbf{a})\right].$$

In particular, if φ has the mean value property then it is harmonic.

Proof. For a smooth function φ consider the function $G(r)$ defined by

$$G(r) = \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r}\varphi(\mathbf{x})\,\mathrm{d}S - \varphi(\mathbf{a}).$$

So G measures the extent by which φ differs from its average. We have already shown

$$G'(r) = F'(r) = \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|<r}\nabla^2\varphi\,\mathrm{d}V,$$

where F was in the proof of the previous result. Obviously this vanishes if φ is harmonic. Note that

$$\begin{aligned}\int_{|\mathbf{x}-\mathbf{a}|<r}\nabla^2\varphi(\mathbf{x})\,\mathrm{d}V &= \nabla^2\varphi(\mathbf{a})\int_{|\mathbf{x}-\mathbf{a}|<r}\mathrm{d}V + \int_{|\mathbf{x}-\mathbf{a}|<r}\left(\nabla^2\varphi(\mathbf{x}) - \nabla^2\varphi(\mathbf{a})\right)\mathrm{d}V \\ &= \frac{4\pi r^3}{3}\nabla^2\varphi(\mathbf{a}) + o\left(r^3\right).\end{aligned}$$

The latter estimate comes from the fact

$$\begin{aligned}\left|\int_{|\mathbf{x}-\mathbf{a}|<r}\left(\nabla^2\varphi(\mathbf{x}) - \nabla^2\varphi(\mathbf{a})\right)\mathrm{d}V\right| &\le \max_{|\mathbf{x}-\mathbf{a}|<r}|\nabla^2\varphi(\mathbf{x}) - \nabla^2\varphi(\mathbf{a})|\int_{|\mathbf{x}-\mathbf{a}|<r}\mathrm{d}V \\ &= \max_{|\mathbf{x}-\mathbf{a}|<r}|\nabla^2\varphi(\mathbf{x}) - \nabla^2\varphi(\mathbf{a})|\left(\frac{4\pi r^3}{3}\right),\end{aligned}$$

and the latter term is $o(r^3)$ owing to the continuity of $\nabla^2\varphi$. In particular, we have that

$$G'(r) = \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|<r}\nabla^2\varphi\,\mathrm{d}V = \frac{r}{3}\nabla^2\varphi(\mathbf{a}) + o(r).$$

Comparing the right hand side with the Taylor expansion

$$G'(r) = G'(0) + rG''(0) + o(r)$$

we deduce

$$G'(0) = 0, \quad G''(0) = \frac{1}{3}\nabla^2\varphi(\mathbf{a}).$$

So for small r we have

$$G(r) = G(0) + rG'(0) + \tfrac{1}{2}r^2G''(0) + o(r^2) = \frac{1}{6}\nabla^2\varphi(\mathbf{a})r^2 + o(r^2).$$

Hence

$$\nabla^2\varphi(\mathbf{a}) = \lim_{r\to 0}\left[\frac{6}{r^2}G(r)\right] = \lim_{r\to 0}\frac{6}{r^2}\left[\frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r}\varphi(\mathbf{x})\,\mathrm{d}S - \varphi(\mathbf{a})\right].$$

If φ satisfies the mean value property then $G(r) = 0$ and so $\nabla^2\varphi$ vanishes. ■

This result tells us that the Laplacian of a function at $\mathbf{x} = \mathbf{a}$ measures the extent to which a function differs from its average value on an infinitesimal sphere centred at $\mathbf{x} = \mathbf{a}$. We see that if a function at $\mathbf{x} = \mathbf{a}$ is *equal* to its average values on any sphere centred at $\mathbf{x} = \mathbf{a}$, then it is necessarily harmonic.

This mean value property gives a simple proof of what called *Liouville's theorem*: a function that is bounded and harmonic on all of $\mathbf{R}^3$ is constant. See example sheet 3.

Harmonic functions also enjoy the following *maximum principle.*

Proposition. *If φ is harmonic in $\Omega \subset \mathbf{R}^3$ then φ cannot have a maximum at any interior point of Ω unless φ is constant.*

Proof. Suppose $\mathbf{a} \in \Omega$ is a point at which $\varphi(\mathbf{a}) \geq \varphi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. First we show that φ must be constant in a ball surrounding $\mathbf{a}$. We have

$$\varphi(\mathbf{a}) \geq \varphi(\mathbf{x}) \quad \text{on} \quad 0 < |\mathbf{x}-\mathbf{a}| \leq \epsilon$$

for some $\epsilon > 0$. We can take ϵ smaller if needs be so the ball $|\mathbf{x}-\mathbf{a}| \leq \epsilon$ is contained in the domain Ω. However, the mean value theorem gives us

$$\varphi(\mathbf{a}) = \frac{1}{4\pi\epsilon^2}\int_{|\mathbf{x}-\mathbf{a}|=\epsilon}\varphi(\mathbf{x})\,\mathrm{d}S \quad \text{i.e.} \quad 0 = \frac{1}{4\pi\epsilon^2}\int_{|\mathbf{x}-\mathbf{a}|=\epsilon}[\varphi(\mathbf{a}) - \varphi(\mathbf{x})]\,\mathrm{d}S.$$

Since the integrand is non-negative, we conclude that $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ on $|\mathbf{x}-\mathbf{a}| = \epsilon$. We can apply the same argument to any smaller sphere centred at $\mathbf{a}$, $|\mathbf{x}-\mathbf{a}| = \epsilon' < \epsilon$, so we conclude $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ on the ball $|\mathbf{x}-\mathbf{a}| < \epsilon$.

Now pick any other point $\mathbf{y} \in \Omega$. Connect $\mathbf{a}$ to $\mathbf{y}$ with a bunch of overlapping balls such that the center of the $(n+1)$th ball is contained in the nth. On the first ball we know $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$. On the center of the second ball we must also have $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$, and so this holds all throughout the second ball by our previous argument. Continuing inductively we get that $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ on a ball containing $\mathbf{y}$. So $\varphi(\mathbf{a}) = \varphi(\mathbf{y})$, and since $\mathbf{y} \in \Omega$ is arbitrary we conclude that φ is constant throughout Ω[18]. ■

We have an immediate corollary: if φ is harmonic on Ω then

$$\varphi(\mathbf{x}) \leq \max_{\mathbf{x}\in\partial\Omega}\varphi(\mathbf{x}).$$

[18]There is a little concern about whether or not we'd have to keep making the balls smaller and smaller, so much so that we could never get to $\mathbf{y}$. This issue can be dealt with once you have learnt about *compactness* in Metric & Topological spaces. Our proof can be made rigorous with a couple of easy tweaks.

This follows from the fact that the maximum of φ on $\Omega \cup \partial\Omega$ *cannot* be obtained in Ω unless φ is constant, in which case the result is obvious.

Harmonic functions also have other nice properties. For example, if φ is harmonic on $\Omega \subset \mathbf{R}^3$ then it is *infinitely differentiable*. You can prove this in Part III. Even more, you can show that a harmonic function will coincide with its Taylor series (analytic). So solutions to Laplace's equation are always very nice functions.

7.6. **Discrete Laplacian and Liouville's theorem (non-examinable).** We've seen that the Laplacian of a function at $\mathbf{x} = \mathbf{a}$ can be thought of as measure of how much a function differs from its average on points around $\mathbf{a}$. A natural definition of the Laplacian of a function on $\mathbf{Z}^n$ would be

$$\nabla^2\varphi(\mathbf{d}) := \frac{\sum_{i=1}^n \varphi(\mathbf{d}+\mathbf{e}_i) + \sum_{i=1}^n \varphi(\mathbf{d}-\mathbf{e}_i)}{2n} - \varphi(\mathbf{d}),$$

where $\mathbf{e}_i$ are the standard Cartesian basis vectors. For example, on $\mathbf{Z}^2$ we have

$$\nabla^2\varphi(i,j) = \frac{\varphi(i+1,j)+\varphi(i,j+1)+\varphi(i-1,j)+\varphi(i,j-1)}{4} - \varphi(i,j), \quad (i,j) \in \mathbf{Z}^2.$$

Liouville's theorem for the discrete Laplcian is as follows.

Theorem (Liouville)**.** *If $\varphi : \mathbf{Z}^n \to \mathbf{Z}$ is bounded and $\nabla^2\varphi = 0$ then $\varphi = \text{const}$.*

Proof. If φ is bounded, then the set $\{\varphi(\mathbf{d}), \mathbf{d} \in \mathbf{Z}^n\}$ is finite and has a least element, α say. Suppose $\varphi(\mathbf{d}) = \alpha$. Then either all the $\varphi(\mathbf{d} \pm \mathbf{e}_i)$ are the same, or at least one is less than α since $\nabla^2\varphi = 0$. The latter can't happen, so we must have $\varphi(\mathbf{d} \pm \mathbf{e}_i) = \alpha$. Carrying on in this fashion we see that $\varphi(\mathbf{d}) = \alpha$ for each $\mathbf{d} \in \mathbf{Z}^n$. ■

Difficult exercise for the very keen: does the statement of the theorem remain true if $\varphi : \mathbf{Z}^n \to \mathbf{Z}$ is replaced with $\varphi : \mathbf{Z}^n \to \mathbf{R}$?

8. Cartesian Tensors

Throughout this section we will be solely dealing with Cartesian coordinate systems. Before we give a definition of a general Cartesian tensor of rank n on $\mathbf{R}^3$, it will be helpful to first build some intuition.

8.1. **A closer look at vectors.** Let $\{\mathbf{e}_i\}$ be a set of right-handed orthonormal basis vectors with respect to some fixed Cartesian axes. When we write a vector as

$$\mathbf{x} = x_i \mathbf{e}_i$$

it is often tempting to identify the *vector* $\mathbf{x}$ with the *components* x_i. However, this would be wrong! Suppose someone else comes along and wants to describe the same vector, but with respect to their favourite set of right-handed Cartesian axes, which are different to yours but coincide at the origin. Denoting the basis vectors with respect to their Cartesian axes by $\{\mathbf{e}'_i\}$, then *the same vector* will be described via

$$\mathbf{x} = x'_i \mathbf{e}'_i.$$

Let us see how the components x_i and x'_i are related. We know

$$x_j \mathbf{e}_j = x'_j \mathbf{e}'_j. \tag{$\star$}$$

Since both $\{\mathbf{e}_j\}$ and $\{\mathbf{e}'_j\}$ are an orthonormal set of vectors, we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{and} \quad \mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}.$$

Dotting both sides of ($\star$) with $\mathbf{e}'_i$ we find

$$x'_i = \delta_{ij} x'_j = (\mathbf{e}'_i \cdot \mathbf{e}'_j) x'_j = \mathbf{e}'_i \cdot (\mathbf{e}'_j x'_j) = (\mathbf{e}'_i \cdot \mathbf{e}_j) x_j.$$

If we define $R_{ij} = (\mathbf{e}'_i \cdot \mathbf{e}_j)$, then we have the simple relation

$$x'_i = R_{ij} x_j. \tag{$\dagger$}$$

Equivalently, if we instead dot both sides of ($\star$) with $\mathbf{e}_i$ we would have found

$$x_i = \delta_{ij} x_j = (\mathbf{e}_i \cdot \mathbf{e}_j) x_j = (\mathbf{e}_i \cdot \mathbf{e}'_j) x'_j = (\mathbf{e}'_j \cdot \mathbf{e}_i) x'_j,$$

i.e.

$$x_i = R_{ji} x'_j \quad \text{or equivalently} \quad x_j = R_{kj} x'_k.$$

If we now use the relation ($\dagger$) in the form $x'_j = R_{jk} x_k$ we get

$$x_i = R_{ji} R_{jk} x_k \quad \text{i.e.} \quad (\delta_{ik} - R_{ji} R_{jk}) x_k = 0.$$

Since this is true *for all* choices of the numbers (x_1, x_2, x_3) we conclude

$$R_{ij} R_{kj} = \delta_{ik}.$$

If we let R denote the matrix with entries R_{ij} then this reads $\mathsf{R}\mathsf{R}^t = I$. So R_{ij} are the components of an *orthogonal matrix*. Since

$$x_j \mathbf{e}_j = x'_i \mathbf{e}'_i = R_{ij} x_j \mathbf{e}'_i$$

must hold for all choices of x_j, we find that the basis vectors are related via

$$\mathbf{e}_j = R_{ij} \mathbf{e}'_i.$$

In particular, since both systems are right handed

$$1 = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = R_{i1}R_{j2}R_{k3}\,\mathbf{e}'_i \cdot (\mathbf{e}'_j \times \mathbf{e}'_k) = R_{i1}R_{j2}R_{k3}\epsilon_{ijk} = \det(\mathsf{R}),$$

where we used $\mathbf{e}'_i \cdot (\mathbf{e}'_j \times \mathbf{e}'_k) = \epsilon_{ijk}$ and the determinant formula you've seen for 3×3 matrices in IA Vectors & Matrices in Michaelmas

$$\det(\mathsf{A}) = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}.$$

So the matrix R with entries R_{ij} is orthogonal with $\det(\mathsf{R}) = 1$. So it is a *rotation*.

Moral of the story: If we transform from one set of right-handed Cartesian basis vectors $\{\mathbf{e}_i\}$ to another $\{\mathbf{e}'_i\}$ then the components of a vector $\mathbf{v}$ will transform according to

$$\boxed{v'_i = R_{ij}v_j}$$

where $R_{ij} = (\mathbf{e}'_i \cdot \mathbf{e}_j)$ are the components of an rotation matrix R, i.e. $\mathsf{R}^t\mathsf{R} = \mathsf{R}\mathsf{R}^t = I$ and $\det(\mathsf{R}) = 1$. Objects whose components transform in this way are called ***vectors*** or *rank 1 Cartesian tensors*, or just *rank 1 tensors* for short. If you prefer a more abstract point of view: you can think of a rank 1 tensor as an equivalence class $[v_i]$, each element of which corresponds to the elements of the vector $\mathbf{v}$ in a given basis. Elements in the same equivalence class are related via a transformation of the above form.

8.2. **A closer look at scalars.** Consider the dot product

$$\sigma = \mathbf{a} \cdot \mathbf{b}$$

for two fixed vectors $\mathbf{a}, \mathbf{b}$. This should be entirely independent of how we describe the basis vectors $\mathbf{a}$ and $\mathbf{b}$, since it has a geometric interpretation in terms of the lengths $|\mathbf{a}|, |\mathbf{b}|$ and the angle between them. Let's investigate this. In terms of the basis vectors $\{\mathbf{e}_i\}$ with $\mathbf{a} = a_i\mathbf{e}_i$ etc. we have

$$\sigma = a_ib_j(\mathbf{e}_i \cdot \mathbf{e}_j) = a_ib_j\delta_{ij} = a_ib_i.$$

If we instead used the basis $\{\mathbf{e}'_i\}$ we would have found

$$\sigma' = a'_ib'_i,$$

where we've used σ' to denote the dot product, because we don't yet know if it will be the same in this basis. Using $a'_i = R_{ip}a_p$ and $b'_i = R_{iq}b_q$ we get

$$\sigma' = R_{ip}R_{iq}a_pb_q = \delta_{pq}a_pb_q = a_pb_p = \sigma.$$

We call objects that transform like this *scalars*, they do not change when we transform one set of right-handed Cartesian basis vectors $\{\mathbf{e}_i\}$ to another $\{\mathbf{e}'_i\}$.

Moral of the story: An object that transforms as

$$\boxed{\sigma' = \sigma}$$

when we change from one right-handed set of Cartesian basis vectors $\{\mathbf{e}_i\}$ to another $\{\mathbf{e}'_i\}$ is called a ***scalar*** or *rank 0 tensor*. The abstract point of view in terms of equivalence classes is dull here: each $[\sigma]$ contains only one element.

8.3. **A closer look at linear maps.** Let $\mathbf{n}$ be a fixed unit vector in $\mathbf{R}^3$ and consider the linear map $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$T : \mathbf{x} \mapsto \mathbf{y} = T(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}.$$

Note that the definition of T does not depend on what basis we're using. The map T gives an orthogonal projection of the vector $\mathbf{x}$ into the plane with normal $\mathbf{n}$. Note that it really is a linear map

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{x} + \beta\mathbf{y} - \alpha(\mathbf{x} \cdot \mathbf{n})\mathbf{n} - \beta(\mathbf{y} \cdot \mathbf{n})\mathbf{n} = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$ and scalars α, β. Suppose we choose to use a Cartesian basis vectors $\{\mathbf{e}_i\}$. Then we can write $\mathbf{x} = x_i\mathbf{e}_i$, $\mathbf{y} = y_i\mathbf{e}_i$ and $\mathbf{n} = n_i\mathbf{e}_i$, so our definition gives

$$y_i\mathbf{e}_i = T(\mathbf{x}) = x_j T(\mathbf{e}_j) = x_j\left(\mathbf{e}_j - n_j n_i \mathbf{e}_i\right) = (\delta_{ij} - n_i n_j)x_j\mathbf{e}_i$$

Set $T_{ij} = \delta_{ij} - n_i n_j$. These are referred to as the components of the linear map T with respect to the basis $\{\mathbf{e}_i\}$. So that we've found

$$y_i = T_{ij}x_j$$

If we did exactly the same thing but with a different set of orthonormal right-handed basis vectors $\{\mathbf{e}'_i\}$, for which $\mathbf{x} = x'_i\mathbf{e}'_i$, etc. we would have found

$$y'_i = T'_{ij}x'_j,$$

where $T'_{ij} = \delta_{ij} - n'_i n'_j$. Using $n'_i = R_{ij}n_j$ gives

$$\begin{aligned} T'_{ij} &= \delta_{ij} - R_{ip}R_{jq}n_p n_q \\ &= R_{ip}R_{jq}(\delta_{pq} - n_p n_q) \\ &= R_{ip}R_{jq}T_{pq} \end{aligned}$$

So we see the components of the linear map T transform according to[19]

$$\boxed{T'_{ij} = R_{ip}R_{jq}T_{pq}}$$

Objects whose components transform in this way are called *rank 2 Cartesian tensors*, or simply *rank 2 tensors* for short[20]. Again, if you like to think in terms of more abstract definitions, you can think of a rank 2 tensor as an equivalence class $[T_{ij}]$, each element of which corresponds to the components of a tensor T in a given basis. Elements of the same equivalence class are related via a transformation of the above form.

[19] If T denotes the matrix with entries T_{ij} etc, then the astute amongst you will have realised this formula just says $\mathsf{T}' = \mathsf{R}\mathsf{T}\mathsf{R}^t$, which is exactly how a matrix transforms under a change of basis generated by an orthogonal transformation, i.e. $\mathsf{R}^t = \mathsf{R}^{-1}$.

[20] Some authors like to call these ***dyads***, but we start to run out of words as the rank of the tensors increase, so we usually just call them rank 2 tensors.

8.4. **Cartesian Tensors of rank n.** We can now describe a general Cartesian tensor of rank n.

Definition. A *tensor* of rank n has components $T_{ij\cdots k}$ (with n indices) with respect to each Cartesian basis $\{\mathbf{e}_i\}$ that transform as

$$\boxed{T'_{ij\cdots k} = R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r}}$$

upon changing from from one right-handed Cartesian basis to another. Here the R_{ij} represent components of an rotation matrix, so in particular $R_{ip}R_{jp} = \delta_{ij}$.

We will be fast and loose with some of our language: we will refer to things like $T_{ij\cdots k}$ as a tensor, rather than the components of a tensor, knowing that these things will change depending on what basis we choose.

Example. If $u_i, v_j, \ldots, w_k$ are the components of n vectors, then

$$T_{ij\cdots k} = u_i v_j \cdots w_k$$

define the components of an nth rank tensor. We can check:

$$T'_{ij\cdots k} = u'_i v'_j \cdots w'_k = R_{ip}R_{jq}\cdots R_{kr}u_p v_q \cdots w_r = R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r}$$

Example. The Kronecker delta δ_{ij} has been defined without any reference to any basis

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

So by definition it is the same, no matter what basis we happen to be using. Hence

$$\delta'_{ij} = \delta_{ij}.$$

But we also see that

$$R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = \delta_{ij} = \delta'_{ij}.$$

Hence $\delta'_{ij} = R_{ip}R_{jq}\delta_{pq}$, so δ_{ij} really is a rank 2 tensor.

Example. We have also defined the Levi Civita symbol without reference to any basis

$$\epsilon_{ijk} = \begin{cases} +1, & (ijk) \text{ an even permutation of } (123), \\ -1, & (ijk) \text{ an odd permutation of } (123), \\ 0, & \text{otherwise.} \end{cases}$$

So by definition $\epsilon_{ijk} = \epsilon'_{ijk}$. But we see that

$$R_{ip}R_{jq}R_{kr}\epsilon_{pqr} = \det(R)\epsilon_{ijk} = \epsilon_{ijk} = \epsilon'_{ijk}.$$

Hence $\epsilon'_{ijk} = R_{ip}R_{jq}R_{kr}\epsilon_{pqr}$ really is a rank 3 tensor.

Example. Experimental evidence suggests that there is a linear relationship between the current $\mathbf{J}$ produced in a given medium that is subject to an electric field $\mathbf{E}$. Such a relationship can be written $\mathbf{J} = \sigma\mathbf{E}$, or in suffix notation

$$J_i = \sigma_{ij}E_j$$

for some array of numbers $\{\sigma_{ij}\}$. Let's show that this thing is actually a tensor: known as the electrical conductivity tensor. Since **E** and **J** are vectors, if we change our coordinate frame we will find[21]

$$\sigma'_{ij}E'_j = J'_i = R_{ip}J_p = R_{ip}\sigma_{pq}E_q = R_{ip}\sigma_{pq}R_{jq}E'_j,$$

and since this must hold for *all* E'_j we conclude

$$\sigma'_{ij} = R_{ip}R_{jq}\sigma_{pq},$$

so σ is a second rank tensor. This is an example of the quotient theorem, which we will prove in a few lectures.

Example. Not all things are tensors. Suppose that for a given Cartesian basis we define the array

$$(A_{ij}) = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \\ 8 & 4 & 7 \end{pmatrix}.$$

And we define $A'_{ij} = 0$ in all other Cartesian bases. Then A_{ij} are not the components of a tensor.

Tensors of the same rank can be added. If A and B are nth rank tensors with components $A_{ij\cdots k}$ and $B_{ij\cdots k}$ (so n indices on both) then we define

$$(A+B)_{ij\cdots k} = A_{ij\cdots k} + B_{ij\cdots k}.$$

If α is a scalar then we define

$$(\alpha A)_{ij\cdots k} = \alpha A_{ij\cdots k}.$$

It is readily seen that both $A+B$ and αA are nth rank tensors. We define the *tensor product* of an mth rank tensor U and an nth rank tensor V by

$$(U \otimes V)_{ij\cdots kpq\cdots r} = U_{ij\cdots k}V_{pq\cdots r},$$

where

$$\underbrace{ij\cdots k}_{m \text{ indices}} \quad \underbrace{pq\cdots r}_{n \text{ indices}}$$

This defines a tensor of rank $n+m$, since

$$\begin{aligned}(U \otimes V)'_{ij\cdots kpq\cdots r} &= U'_{ij\cdots k}V'_{pq\cdots r} \\ &= R_{ia}R_{jb}\cdots R_{kc}U_{ab\cdots c}R_{pd}R_{qe}\cdots R_{rf}V_{de\cdots f} \\ &= R_{ia}R_{jb}\cdots R_{kc}R_{pd}R_{qe}\cdots R_{rf}(U\otimes V)_{ab\cdots cde\cdots f}.\end{aligned}$$

Given an nth rank tensor with components $T_{ijk\cdots l}$ we can define an $(n-2)$th rank tensor by *contracting* on two of the indices. For instance, contracting on i and j is defined by

$$\delta_{ij}T_{ijk\cdots l} = T_{iik\cdots l}.$$

This really is a tensor of rank $n-2$, since

$$T'_{iik\cdots l} = R_{ip}R_{iq}R_{kr}\cdots R_{ls}T_{pqr\cdots s} = \delta_{pq}R_{kr}\cdots R_{ls}T_{pqr\cdots s} = R_{kr}\cdots R_{ls}T_{ppr\cdots s}.$$

We can choose to contract on any pair of indices we like, not just the first two.

[21]Here we use $E'_j = R_{jq}E_q \Leftrightarrow E_q = R_{jq}E'_j$, which just reflects the fact that $\mathsf{R}^{-1} = \mathsf{R}^t$ for $\mathsf{R} \in SO(3)$.

We say a tensor $T_{ij\cdots k}$ is *symmetric* in (i,j) if

$$T_{ij\cdots k} = T_{ji\cdots k}.$$

This is a well-defined property of the tensor, i.e. it doesn't matter which coordinate frame choose, since if $T_{ij\cdots k}$ is symmetric in (i,j) then

$$T'_{ij\cdots k} = R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r} = R_{ip}R_{jq}\cdots R_{kr}T_{qp\cdots r} = R_{iq}R_{jp}\cdots R_{kr}T_{pq\cdots r} = T'_{ji\cdots k}$$

i.e. $T'_{ij\cdots k}$ is also symmetric in (i,j). Similarly we say $A_{ij\cdots k}$ is anti-symmetric in (i,j) if $T_{ij\cdots k} = -T_{ji\cdots k}$. We could be talking about any particular pair of indices here – it doesn't necessarily need to be the first and second. We say a tensor is *totally* symmetric (anti-symmetric) if it is symmetric (anti-symmetric) in each pair of indices.

Example. The tensors δ_{ij} and $a_ia_ja_k$ are totally symmetric tensors of rank two and three respectively. ϵ_{ijk} is totally anti-symmetric.

In fact, the only totally anti-symmetric tensor on $\mathbf{R}^3$ of rank $n = 3$ is proportional to ϵ_{ijk}, and there aren't any (non-zero) ones of higher rank. Suppose $T_{ij\cdots k}$ is a totally anti-symmetric tensor of rank n. Then $T_{ij\cdots k} = 0$ if any two indices are the same. If $n > 3$ then there will always be at least two matching indices in $T_{ij\cdots k}$, by the pigeonhole principle. When $n = 3$, there are only only $3! = 6$ non-zero components, and if

$$T_{123} = T_{231} = T_{312} = \lambda$$

then $T_{213} = T_{321} = T_{132} = -\lambda$ by anti-symmetry. Hence $T_{ijk} = \lambda\epsilon_{ijk}$.

8.5. **Tensor calculus.** Remember that a vector field is just a function that gives you a vector $\mathbf{v}(\mathbf{x})$ at each point $\mathbf{x} \in \mathbf{R}^3$. Said differently, a vector field gives you a rank 1 tensor at each $\mathbf{x} \in \mathbf{R}^3$. Similarly, a scalar field $\varphi(\mathbf{x})$ gives a scalar, or rank 0 tensor at each point. We define a tensor field of rank n, $T_{ij\cdots k}(\mathbf{x})$ so that for each $\mathbf{x}_0 \in \mathbf{R}^3$, $T_{ij\cdots k}(\mathbf{x}_0)$ define the components of an nth rank Cartesian tensor.

Recall that Cartesian coordinates transform as

$$x'_i = R_{ij}x_j \quad \Leftrightarrow \quad x_j = R_{ij}x'_i.$$

Differentiating both sides with respect to x'_k we find

$$\frac{\partial x_j}{\partial x'_k} = R_{ij}\frac{\partial x'_i}{\partial x'_k} = R_{ij}\delta_{ik} = R_{kj}.$$

So the chain rule gives

$$\frac{\partial}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{1}{\partial x_j} = R_{ij}\frac{\partial}{\partial x_j}.$$

This gives us a good way of understanding how tensor fields change when we differentiate them. Note that the rotation matrix is *constant*.

Proposition. *If $T_{ij\cdots k}(\mathbf{x})$ is a tensor field of rank n, then*

$$\left(\frac{\partial}{\partial x_p}\right)\left(\frac{\partial}{\partial x_q}\right)\cdots\left(\frac{\partial}{\partial x_r}\right)T_{ij\cdots k}(\mathbf{x})$$

(with m derivatives in total) is a tensor field of rank $m + n$.

Proof. Denote the quantity by $A_{pq\cdots rij\cdots k}$ so that

$$\begin{aligned} A'_{pq\cdots rij\cdots k} &= \left(\frac{\partial}{\partial x'_p}\right)\left(\frac{\partial}{\partial x'_q}\right)\cdots\left(\frac{\partial}{\partial x'_r}\right) T'_{ij\cdots k}(\mathbf{x}) \\ &= R_{pa}R_{qb}\cdots R_{rc}\left(\frac{\partial}{\partial x_a}\right)\left(\frac{\partial}{\partial x_b}\right)\cdots\left(\frac{\partial}{\partial x_c}\right) R_{id}R_{je}\cdots R_{kf}T_{de\cdots f}(\mathbf{x}) \\ &= R_{pa}R_{qb}\cdots R_{rc}R_{id}R_{je}\cdots R_{kf}A_{ab\cdots cde\cdots f} \end{aligned}$$

which is the transformation law for a rank $n+m$ tensors. ∎

Example. For a scalar field $\varphi = \varphi(\mathbf{x})$, consider the gradient $\nabla\varphi(\mathbf{x})$, or in components

$$[\nabla\varphi]_i = \frac{\partial\varphi}{\partial x_i}.$$

By the previous result, these are the components of a rank $0+1=1$ tensor field, or simply a vector field.

Example. For a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ we have the divergence

$$\nabla\cdot\mathbf{v} = \frac{\partial v_i}{\partial x_i}.$$

Not unsurprisingly, this transforms as a scalar field

$$\frac{\partial v'_i}{\partial x'_i} = R_{ip}\frac{\partial}{\partial x_p}R_{iq}v_q = R_{ip}R_{iq}\frac{\partial v_q}{\partial x_p} = \delta_{pq}\frac{\partial v_q}{\partial x_p} = \frac{\partial v_p}{\partial x_p}.$$

Example. For a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ consider the curl $\nabla\times\mathbf{v}$. Then in components

$$[\nabla\times\mathbf{v}]_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}.$$

This transforms as a vector field, but it is not immediately obvious. We have

$$\begin{aligned} \epsilon'_{ijk}\frac{\partial v'_k}{\partial x'_j} &= R_{ia}R_{jb}R_{kc}\epsilon_{abc}R_{jp}\frac{\partial}{\partial x_p}R_{kq}v_q \\ &= R_{ia}\epsilon_{abc}R_{jb}R_{jp}R_{kc}R_{kq}\frac{\partial v_q}{\partial x_p} \\ &= R_{ia}\epsilon_{abc}\delta_{bp}\delta_{cq}\frac{\partial v_q}{\partial x_p} \\ &= R_{ia}\epsilon_{abc}\frac{\partial v_c}{\partial x_b}, \end{aligned}$$

so $[\nabla\times\mathbf{v}]_i$ does indeed transform like a vector field.

We can integrate tensor fields just as we can integrate vector fields. Recall the divergence theorem for vector fields, written using suffix notation

$$\int_V \frac{\partial v_i}{\partial x_i}\,\mathrm{d}V = \int_{\partial V} v_i n_i\,\mathrm{d}S,$$

where $\mathbf{n}$ is the *outward* normal to the surface enclosing V. An analogous result holds for tensor fields.

Proposition. *For a smooth tensor field $T_{ij\cdots k\cdots l}(\mathbf{x})$ we have*

$$\int_V \frac{\partial T_{ij\cdots k\cdots l}}{\partial x_k}\,\mathrm{d}V = \int_{\partial V} T_{ij\cdots k\cdots l}n_k\,\mathrm{d}S.$$

Note that whatever $\frac{\partial}{\partial x_k}$ appears on the left is replaced with n_k on the right.

Proof. Apply the usual divergence theorem to the vector field[22]

$$v_k \coloneqq a_i b_j \cdots c_l T_{ij\cdots k\cdots l}$$

where $a_i, b_j, \ldots, c_k$ are components of constant vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c}$. On the right hand side the only free index is k, since all of the others have been contracted with the components of one of the vectors $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{c}$. This gives

$$\int_V \frac{\partial v_k}{\partial x_k} \,\mathrm{d}V = a_i b_j \cdots c_l \int_V \frac{\partial T_{ij\cdots k\cdots l}}{\partial x_k} \,\mathrm{d}V$$

and

$$\int_{\partial V} v_k n_k \,\mathrm{d}S = a_i b_j \cdots c_l \int_{\partial V} T_{ij\cdots k\cdots l} n_k \,\mathrm{d}S.$$

The result follows because the $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c}$ were arbitrary. Indeed, suppose we wanted to check the result for when all the free indices were equal to 1. Then we would set

$$a_i = \delta_{i1}, \quad b_j = \delta_{j1}, \quad \cdots \quad c_k = \delta_{k1},$$

so that both sides would reduce to

$$\int_V \frac{\partial T_{11\cdots k\cdots 1}}{\partial x_k} \,\mathrm{d}V = \int_{\partial V} T_{11\cdots k\cdots 1} n_k \,\mathrm{d}S.$$

A similar idea can be used for any other choice of the free indices. ∎

8.6. **Rank 2 tensors.** Particularly common in applied mathematics and physics are tensors of rank two. For any rank 2 tensor T_{ij} we can always write

$$\begin{aligned} T_{ij} &= \frac{1}{2}\left(T_{ij} + T_{ji}\right) + \frac{1}{2}\left(T_{ij} - T_{ji}\right) \\ &\equiv S_{ij} + A_{ij}. \end{aligned}$$

We see that S_{ij} is *symmetric*

$$S_{ij} = \frac{1}{2}\left(T_{ji} + T_{ij}\right) = S_{ji}$$

whereas A_{ij} is *anti-symmetric*

$$A_{ij} = -\frac{1}{2}\left(T_{ji} - T_{ij}\right) = -A_{ji}.$$

The original tensor has $3^2 = 9$ independent components. The components of a symmetric tensor S_{ij} are completely determined by the entries with $i \geq j$, of which there are

$$1 + 2 + 3 = 6$$

components. Whereas an anti-symmetric tensor A_{ij} is completely determined by the components with $i > j$ (since the $i = j$ components are necessarily zero), of which there are

$$1 + 2 = 3.$$

This is good, because $6 + 3 = 9$. Since the anti-symmetric tensor A_{ij} is completely determined by 3 of its components, we expect that all the information contained in A_{ij} can be somehow encoded into a vector ω_i.

[22]This really does define a vector field. First note that $a_p b_q \cdots c_r T_{ij\cdots k\cdots l}$ are the components of the tensor $\mathbf{a} \otimes \mathbf{b} \otimes \cdots \otimes \mathbf{c} \otimes T$, and if T has rank $n+1$ then this tensor has rank $2n+1$. We then contract on the indices $(p, i), (q, j), \ldots (r, l)$, reducing the rank of the tensor by $2n$ and leaving only one free index k.

Proposition. *Every rank two tensor can be decomposed uniquely as*

$$T_{ij} = S_{ij} + \epsilon_{ijk}\omega_k$$

where the vector $\boldsymbol{\omega}$ *is defined by* $\omega_i = \frac{1}{2}\epsilon_{ijk}T_{jk}$ *and* S_{ij} *is symmetric.*

Proof. Given our previous discussion, we can identify

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right),$$

and it remains to show that $\epsilon_{ijk}\omega_k = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$. We have

$$\begin{aligned}\epsilon_{ijk}\omega_k &= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{klm}T_{lm}\\ &= \tfrac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)T_{lm}\\ &= \tfrac{1}{2}\left(T_{ij} - T_{ji}\right).\end{aligned}$$

If there were two such decompositions

$$S_{ij} + \epsilon_{ijk}\omega_k = \tilde{S}_{ij} + \epsilon_{ijk}\tilde{\omega}_k$$

then taking symmetric parts of both sides yields $S_{ij} = \tilde{S}_{ij}$, and so $\epsilon_{ijk}\omega_k = \epsilon_{ijk}\tilde{\omega}_k$. The latter implies $\omega_k = \tilde{\omega}_k$, so the decomposition is unique. ∎

Example. Suppose that each point $\mathbf{x}$ in an elastic body undergoes a small displacement of $\mathbf{u}(\mathbf{x})$. Then two nearby points that were initially separated by $\delta\mathbf{x}$ become separated by

$$(\mathbf{x} + \delta\mathbf{x} + \mathbf{u}(\mathbf{x} + \delta\mathbf{x})) - \mathbf{x} - \mathbf{u}(\mathbf{x}) = \delta\mathbf{x} + \mathbf{u}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{u}(\mathbf{x}).$$

So the *change in displacement* between the two points in the body is

$$\mathbf{u}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{u}(\mathbf{x}).$$

This gives a measure of how much deformation happens to the body. Using Taylor's theorem with Cartesian coordinates and suffix notation

$$u_i(\mathbf{x} + \delta\mathbf{x}) - u_i(\mathbf{x}) = \frac{\partial u_i}{\partial x_j}\delta x_j + o(\delta\mathbf{x}).$$

We decompose the second rank tensor $\partial u_i/\partial x_j$ as

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \epsilon_{ijk}\omega_k$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is called the *linear strain tensor* and

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\frac{\partial u_j}{\partial x_k} = -\frac{1}{2}(\nabla \times \mathbf{u})_i.$$

This tells us that

$$u_i(\mathbf{x} + \delta\mathbf{x}) - u_i(\mathbf{x}) = e_{ij}\delta x_j + [\boldsymbol{\omega} \times \delta\mathbf{x}]_i + o(\delta\mathbf{x}).$$

The second term is just a solid body rotation[23] whereas the first term tells us about how the material stretches and/or compresses internally.

[23]Remember from IA Vectors & Matrices that a rotation about $\mathbf{n}$ by θ is

$$R\mathbf{x} = \mathbf{x}\cos\theta + (\mathbf{n} \times \mathbf{x})\sin\theta + \mathbf{n}(\mathbf{n} \cdot \mathbf{x})(1 - \cos\theta) = \mathbf{x} + \theta(\mathbf{n} \times \mathbf{x}) + o(\theta), \quad \theta \to 0.$$

So the change in displacement owing to a small rotation is $\mathsf{R}\delta\mathbf{x} - \delta\mathbf{x} = \theta(\mathbf{n} \times \delta\mathbf{x}) + o(\theta)$, hence the identification of the second term with a solid rotation.

A well known symmetric rank 2 tensor is the *inertia tensor*. Suppose a body with uniform density $\rho(\mathbf{x})$ occupies a volume V. Suppose each point in the body is rotating with constant angular velocity $\boldsymbol{\omega}$ about the origin, so the velocity of a point at $\mathbf{x} \in V$ is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$. Then the total angular momentum about the origin is

$$\begin{aligned}\mathbf{L} &= \int_V (\mathbf{x} \times \mathbf{v})\rho(\mathbf{x})\,\mathrm{d}V \\ &= \int_V \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x})\rho(\mathbf{x})\,\mathrm{d}V.\end{aligned}$$

Using suffix notation we find

$$L_i = \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \omega_i - x_i x_j \omega_j)\,\mathrm{d}V = I_{ij}\omega_j$$

where we have defined the *inertia tensor* by

$$I_{ij} = \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\,\mathrm{d}V,$$

and the integral is taken over $\mathcal{V} = \{(x_1, x_2, x_3) : \mathbf{x} = x_i\mathbf{e}_i \in V\}$. If we had used a different set of Cartesian basis vectors $\{\mathbf{e}'_i\}$ with $\mathbf{x} = x'_i\mathbf{e}'_i$ etc we would have found

$$I'_{ij} = \int_{\mathcal{V}'} \rho(\mathbf{x})(x'_k x'_k \delta_{ij} - x'_i x'_j)\,\mathrm{d}V = R_{ip}R_{jq}\int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \delta_{pq} - x_p x_q)\,\mathrm{d}V = R_{ip}R_{jq}I_{pq},$$

where $\mathcal{V}' = \{(x'_1, x'_2, x'_3) : \mathbf{x} = x'_i\mathbf{e}'_i \in V\}$. So it really is a symmetric tensor of rank 2.

Example. Consider an ellipsoid given in a particular Cartesian coordinate system by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \le 1,$$

with uniform density ρ_0, so has mass $M = \frac{4\pi abc}{3}\rho_0$. To compute the inertia tensor for this solid we use scaled spherical polar coordinates

$$x_1 = ar\cos\phi\sin\theta, \quad x_2 = br\sin\phi\sin\theta, \quad x_3 = cr\cos\theta,$$

with $0 \le r \le 1$. Note that if $i \ne j$ then

$$\int_V \rho_0 x_i x_j\,\mathrm{d}V = 0,$$

by symmetry. Also

$$\begin{aligned}I_{11} &= \rho_0\int_V (x_2^2 + x_3^2)\,\mathrm{d}V \\ &= \rho_0 abc\int_{r=0}^1\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(b^2\sin^2\phi\sin^2\theta + c^2\cos^2\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{\rho_0 abc}{5}\int_0^{\pi}\left(\pi b^2\sin^2\theta + 2\pi c^2\cos^2\theta\right)\sin\theta\,\mathrm{d}\theta \\ &= \frac{3M}{4}\frac{1}{5}\int_0^{\pi}\left(b^2\sin\theta + (2c^2 - b^2)\cos^2\theta\sin\theta\right)\mathrm{d}\theta \\ &= \frac{3M}{20}\left(2b^2 + \frac{2}{3}(2c^2 - b^2)\right) \\ &= \frac{M}{5}\left(b^2 + c^2\right).\end{aligned}$$

and by symmetry

$$I_{22} = \frac{M}{5}\left(a^2 + c^2\right), \quad I_{33} = \frac{M}{5}\left(a^2 + b^2\right).$$

So in the coordinate system aligned with the axes of symmetry of the ellipsoid

$$(I_{ij}) = \frac{M}{5}\begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}.$$

Note that if $a = b = c$ then $I_{ij} = \frac{2}{5}M\delta_{ij}$.

For symmetric, second rank tensors we also have the following result.

Proposition. *If T_{ij} is a symmetric, second rank tensor then there exists a choice of right-handed Cartesian coordinates axes in which*

$$(T_{ij}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

The corresponding Cartesian axes are called the principal axes for T.

Proof. This is a direct consequence of the fact that a real symmetric matrix can be diagonalised using orthogonal transformation R, for which we can assume $\det(\mathsf{R}) = 1$ without loss of generality. ■

In particular, for the inertia tensor I_{ij} we can *always* choose a set of axes so that the components form a diagonal array. These axes usually correspond to some obvious lines of symmetry on the body.

8.7. **Invariant and isotropic tensors.** We say a tensor is *isotropic* if it is invariant under changes of Cartesian coordinates, i.e.

$$T'_{ij\cdots k} = R_{ip}R_{jq}\cdots R_{kr}T_{pq\cdots r} = T_{ij\cdots k},$$

for *any* rotation matrix R.

Example. We have already seen three types of isotropic tensor:

(i) *Every* tensor if rank 0 is isotropic.

(ii) The Kronecker delta is isotropic

$$\delta'_{ij} = R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = \delta_{ij}.$$

(iii) The Levi-Civita tensor is isotropic

$$\epsilon'_{ijk} = R_{ip}R_{jq}R_{kr}\epsilon_{pqr} = \det(\mathsf{R})\epsilon_{ijk} = \epsilon_{ijk}.$$

We can actually classify all isotropic tensors on $\mathbf{R}^3$. The general result is due to Hermann Weyl, and is given in his book *The Classical Groups*.

Proposition. *Isotropic tensors on $\mathbf{R}^3$ are classified as:*

(a) All tensors of rank 0 are isotropic.

(b) There are no non-zero isotropic tensors of rank 1.

(c) The most general isotropic tensor of rank 2 is $\alpha\delta_{ij}$, with α a scalar.

(d) The most general isotropic tensor of rank 3 is $\beta\epsilon_{ijk}$, with β a scalar.

(e) The most general isotropic tensor of rank 4 is $\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}$, α,β,γ *scalars.*

(f) The most general isotropic tensor of rank $n \geq 4$ *is a linear combination of products of Kronecker deltas and Levi-Civita tensors.*

The full proof of this result is a little too involved for this course. However, we can certainly give the basic idea for rank ≤ 3.

Sketch proof. (a) Rank 0 tensors are isotropic by definition.

(b) If v_i are the components of an isotropic vector then

$$v_i = R_{ij}v_j$$

for *any* rotation. We will use rotations by π about each axis. Take

$$(R_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $v_1 = T_{1j}v_j = -v_1$, so $v_1 = 0$ and $v_2 = T_{2j}v_j = -v_2$ so $v_2 = 0$. Similarly, using

$$(R_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

we find $v_3 = T_{3j}v_j = -v_j$ so $v_3 = 0$, i.e. $v_i = 0$, and this holds in all frames.

(c) If T_{ij} is isotropic then $T_{ij} = R_{ip}R_{jq}T_{pq}$ for any R. We will use rotations by $\pi/2$ about each axis. First we take

$$(R_{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} T_{13} &= R_{1p}R_{3q}T_{pq} = R_{12}R_{33}T_{23} = T_{23}, \\ T_{23} &= R_{2p}R_{3q}T_{pq} = R_{21}R_{33}T_{13} = -T_{13}, \end{aligned}$$

so $T_{13} = T_{23} = 0$. We also have

$$T_{11} = R_{1p}R_{1q}T_{pq} = R_{12}R_{12}T_{22} = T_{22}.$$

Next we take

$$(R_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} T_{32} &= R_{3p}R_{2q}T_{pq} = R_{32}R_{23}T_{23} = -T_{23} = 0, \\ T_{12} &= R_{1p}R_{2q}T_{pq} = R_{11}R_{23}T_{13} = -T_{13} = 0, \\ T_{31} &= R_{3p}R_{1q}T_{pq} = R_{32}R_{11}T_{21} = -T_{21}, \\ T_{21} &= R_{2p}R_{1q}T_{pq} = R_{23}R_{11}T_{31} = T_{31}, \end{aligned}$$

so $T_{31} = T_{21} = 0$. We also have

$$T_{22} = R_{2p}R_{2q}T_{pq} = R_{23}R_{23}T_{33} = T_{33}.$$

In conclusion $T_{ij} = 0$ if $i \neq j$ and $T_{11} = T_{22} = T_{33}$. Hence $T_{ij} = \alpha\delta_{ij}$ for a scalar α.
(d) Similar, just more indices.

■

As an example of the use of isotropic tensors, consider integrals of the form

$$T_{ij\cdots k} = \int_{|\mathbf{x}|<R} f(r)\, x_i x_j \cdots x_k \,\mathrm{d}V(\mathbf{x}),$$

where $|\mathbf{x}| = r$ and $\mathrm{d}V(\mathbf{x}) = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. Then $f(r)$ and the domain $\{\mathbf{x} : |\mathbf{x}| < R\}$ are invariant under rotations. We have

$$\begin{aligned} T'_{ij\cdots k} &= \int_{|\mathbf{x}|<R} f(r)\, x'_i x'_j \cdots x'_k \,\mathrm{d}V(\mathbf{x}) \\ &= \int_{|\mathbf{x}|<R} f(r)\, R_{ip}x_p R_{jq}x_q \cdots R_{kr}x_r \,\mathrm{d}V(\mathbf{x}). \end{aligned}$$

Now make the substitution $y_i = R_{ij}x_j$ for $j = 1, 2, 3$. Since it is a rotation the determinant of this transformation is 1. Hence

$$T'_{ij\cdots k} = \int_{|\mathbf{y}|<R} f(r)\, y_i y_j \cdots y_k \,\mathrm{d}V(\mathbf{y}),$$

where $\mathrm{d}V(\mathbf{y}) = \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3$. Since $\mathbf{y}$ is just a dummy index

$$T'_{ij\cdots k} = \int_{|\mathbf{x}|<R} f(r)\, x_i x_j \cdots x_k \,\mathrm{d}V(\mathbf{x}) = T_{ij\cdots k}.$$

So these are isotropic tensors! Taking $R \to \infty$ corresponds to integrating over all $\mathbf{R}^3$.

Example. Consider the array

$$T_{ij} = \int_{\mathbf{R}^3} e^{-r^5} x_i x_j \,\mathrm{d}V.$$

By the previous discussion, we know that T_{ij} is isotropic, so $T_{ij} = \alpha\delta_{ij}$ for some constant α. Contracting on i and j

$$3\alpha = \int_{\mathbf{R}^3} e^{-r^5} r^2 \,\mathrm{d}V = 4\pi \int_0^\infty r^4 e^{-r^5} \,\mathrm{d}r = \frac{4\pi}{5}.$$

Hence $T_{ij} = \frac{4\pi}{15}\delta_{ij}$.

Example. The inertia tensor for a ball of radius R and uniform density ρ_0 is

$$I_{ij} = \int_{|\mathbf{x}|<R} \rho_0 (x_k x_k \delta_{ij} - x_i x_j) \,\mathrm{d}V.$$

This is the sum of two isotropic tensors, so $I_{ij} = \alpha\delta_{ij}$ for some appropriate constant α. Contracting on i and j

$$3\alpha = \int_{|\mathbf{x}|<R} 2\rho_0 r^2 \,\mathrm{d}V = 2\rho_0 \int_{r=0}^R \int_{\theta=0}^\pi \int_{\phi=0}^{2\pi} r^4 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi = \frac{8\pi R^5 \rho_0}{5} = \frac{6MR^2}{5},$$

where $M = \frac{4\pi}{3}R^3\rho_0$ is the mass of the ball. Hence $I_{ij} = \frac{2}{5}MR^2\delta_{ij}$.

8.8. **Tensors as multi-linear maps and the quotient rule.** For a tensor T_{ij} in a given Cartesian coordinate system, consider the bilinear map $t : \mathbf{R}^3 \times \mathbf{R}^3 \mapsto \mathbf{R}$ defined by

$$t(\mathbf{a}, \mathbf{b}) \coloneqq T_{ij} a_i b_j.$$

This map is well-defined, in the sense that it doesn't depend on what Cartesian basis we use. Indeed, the right hand side is a scalar, so invariant under change of Cartesian basis. So we can define a bilinear map from a rank two tensor.

Conversely, if $t : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ is bilinear map, then for a given Cartesian basis $\{\mathbf{e}_i\}$ it defines an array T_{ij} via

$$t(\mathbf{a}, \mathbf{b}) = a_i b_j t(\mathbf{e}_i, \mathbf{e}_j) \coloneqq T_{ij} a_i b_j.$$

If we use a different Cartesian coordinate system $\{\mathbf{e}'_i\}$ related to the old one via $\mathbf{e}'_i = R_{ip}\mathbf{e}_p$ then, using bilinearity of t, we see that the components of the array transform as

$$T'_{ij} = t(\mathbf{e}'_i, \mathbf{e}'_j) = R_{ip} R_{jq} t(\mathbf{e}_p, \mathbf{e}_q) = R_{ip} R_{jq} T_{pq}.$$

So the array T_{ij} transforms as a rank 2 tensor.

These observations can be used to construct an explicit, one-to-one correspondence between bilinear maps and rank 2 Cartesian tensor. In particular, if the map

$$(\mathbf{a}, \mathbf{b}) \mapsto T_{ij} a_i b_j$$

is genuinely a bilinear map (independent of basis), then the array T_{ij} define a rank 2 tensor. Exactly the same idea works with tensors of higher rank: if the map

$$(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c}) \mapsto T_{ij\ldots k} a_i b_j \cdots c_k$$

genuinely defines a multilinear map (independent of basis), then the array $T_{ij\ldots k}$ transform as a tensor of the appropriate rank.

This gives us a nice way of proving the *quotient theorem*. We saw earlier in this chapter, when looking at the conductivity tensor, that if $\mathbf{E}$ and $\mathbf{J}$ are vectors and $J_i = \sigma_{ij} E_j$, then σ_{ij} were necessarily the components of a rank 2 tensor. The quotient theorem is the same as this, but for arbitrary rank.

Proposition. *Let $T_{i\ldots jp\ldots q}$ is an array of numbers defined in each Cartesian coordinate system such that*

$$v_{i\ldots j} \coloneqq T_{i\ldots jp\ldots q} u_{p\ldots q}$$

is a tensor for each tensor $u_{p\ldots q}$, then $T_{i\ldots jp\ldots q}$ is a tensor.

Proof. Take the special case $u_{p\ldots q} = c_p \cdots d_q$ for vectors $\mathbf{c}, \ldots, \mathbf{d}$. Then

$$v_{i\ldots j} \coloneqq T_{i\ldots jp\ldots q} c_p \cdots d_q$$

is a tensor and in particular

$$v_{i\ldots j} a_i \cdots b_j = T_{i\ldots jp\ldots q} a_i \cdots b_j c_p \cdots d_q$$

is a scalar for each $\mathbf{a}, \ldots, \mathbf{b}, \mathbf{c}, \ldots, \mathbf{d}$. So the right hand side is a scalar (i.e. it is independent of Cartesian basis), and gives rise to a well-defined multi-linear map via

$$t(\mathbf{a}, \ldots, \mathbf{b}, \mathbf{c}, \ldots, \mathbf{d}) \coloneqq T_{i\ldots jp\ldots q} a_i \cdots b_j c_p \cdots d_q.$$

By our previous discussion we conclude that $T_{i\ldots jp\ldots q}$ is a tensor. ∎

Example. We have already seen the *linear strain tensor*

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

which measures the infinitesimal deformation of a body in which each point $\mathbf{x}$ undergoes a displacement of $\mathbf{u}(\mathbf{x})$. Experiment suggests that the *stresses*, or internal forces, experienced by a body depends linearly on the strain at each point. Stresses are measured by the *stress tensor* σ_{ij}. So there should be a collection of $3^4 = 81$ numbers c_{ijkl} such that

$$\sigma_{ij} = c_{ijkl}e_{kl}.$$

We *cannot* immediately apply the quotient theorem, because e_{kl} is not arbitrary: it is necessarily symmetric. However, if we assume $c_{ijkl} = c_{ijlk}$ then we *can* apply the quotient theorem (see example sheet 4). On this assumption, the array c_{ijkl} must correspond to the components of a rank 4 tensor. It is called the *stiffness tensor*, and is a property of the material that is under stress. If the material is isotropic then we should write, for scalars λ, β, γ,

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{kj},$$

so that

$$\begin{aligned}\sigma_{ij} &= \lambda\delta_{ij}e_{kk} + \beta e_{ij} + \gamma e_{ji}\\ &\equiv \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},\end{aligned}$$

where in the final line we used $e_{ij} = e_{ji}$ and defined $2\mu = \beta + \gamma$. This is Hooke's law for an isotropic material. It is the higher dimensional analogue of the familiar equation $F = -kx$ for an elastic spring. We can use it to get the strain tensor in terms of the stress tensor. First contract on i and j so that

$$\sigma_{ii} = (3\lambda + 2\mu)e_{kk}, \quad \text{i.e.} \quad e_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu},$$

assuming $3\lambda \neq -2\mu$. Then we get

$$2\mu e_{ij} = \sigma_{ij} - \left(\frac{\lambda}{3\lambda + 2\mu}\right)\sigma_{kk}\delta_{ij}.$$

If you found this section on tensors and tensor calculus difficult, don't worry – you're in good company. I offer you the following source of comfort:

Do not worry about your difficulties in Mathematics. I can assure you mine are still greater – Albert Einstein.

When Einstein tried to extend his special theory of relativity to include gravity, he had to overcome some major hurdles: amongst them were tensor calculus and its use in differential geometry. He had to seek help from some great mathematicians, notably Tullio Levi-Civita, Hermann Minkowski, Felix Klein and David Hilbert. You will have to make do with the humble Faculty of Mathematics here at Cambridge – good luck!

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