Generating functions for integer sequences defined by the evolution of cellular automata with even rule numbers

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Abstract

This paper presents a unified treatment of generating functions G(t,x) for the evolution of one-dimensional cellular automata (CAs) from single seed initial states. For such CAs, with even rule number R, explicit expressions are given for G(t,x). When x = 1, 2, 10, the expressions that follow for G(t,1), G(t,2), G(t,10) generate

a) the number N(t) of ones in the display line for time step t,

b) the decimal integer B(t) of which the display line for time step t is the binary representation,

c) the decimal integer D(t) of which the display line is itself the denary representation.

A good selection of the results in the special cases are well-known, with citation below, but key results for the more elusive cases R = 62, 126, 182 amongst others are new here. Treatment of most elusive case of all, that of Rule 150, is given in a companion paper, where an analogue is obtained for the case of B(t) for Wolfram's well-known formula, in terms of Jacobsthal numbers, for N(t).

1 Introduction

This paper is the first of a set of three papers which studies various integer sequences defined by each member of a certain set Σ of one-dimensional cellular automata (CAs) [1, 2], and which provides a unified treatment of the generating functions (GFs) of such sequences.

The notation CA(R) is used for the CA with rule number R [1]. A $CA(R) \in \Sigma$, if R is even, and if it produces a display of steadily increasing width. In all cases considered the term display refers to the evolution of the CA from an initial state of single seed [1].

CAs $\in \Sigma$ are of two types: legal [1] or not legal. A CA is legal, nd $\in \Sigma_L$, if its display is symmetric when presented in centre-justified format. Write $\Sigma = \Sigma_L + \Sigma_I$. Here Σ_I contains all CAs $\in \Sigma$ that are not legal except those with CA(30) and CA(110). Various subsets of CAs $\in \Sigma$ give rise to identical displays. For example, the CAs with rule numbers given by

$$18, 146, 218 \in \Sigma_L, \quad \text{and} \quad 26, 82, 154, 210 \in \Sigma_I, \tag{1}$$

all give rise to the same display as does CA(90). This is so even though CA(82) and CA(210) have definitions that are mirror images of those of CA(26) and CA(154). Thus, to treat all distinct, and treatable, displays, it is sufficient to confine attention to the CAs with rule numbers given by

$$22, 50, 54, 90, 94, 126, 150, 182, 222 \in \Sigma_L, \tag{2}$$

and

$$28, 60, 62, 92, 158, 188, 190, 220 \in \Sigma_I. \tag{3}$$

Of these, all but four are quite easy to deal with. Significant effort is needed for the CAs of rule numbers 62, 126, 150, 182, and many of the results that are new in this work are associated with them. In the case of the CA(30) and CA(110), and their mirror images, control of the GFs of their

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integer sequences seems to be well out of reach, which explains why they are excluded from Σ_I here. That this should be the case should come as no surprise if pp27-39 of NKS [2] are consulted.

This paper derives an explicit expression for the general GF $G(t,x)$ of Definition 1 below, for each CA(R) for R $\in \Sigma$, except for CA(150), allowing specialisation to yield results for the sequences $\mathcal{N}$, $\mathcal{B}$, $\mathcal{D}$ defined by Definitions 2 –4, below. While there is no need in some cases to quote explicitly results thereby made available for the GFs $G(t,1)$ and $G(t,2)$ for $\mathcal{N}$, $\mathcal{B}$, it is worth stressing the fact they come directly from a common source simply by the substitutions of $x = 1$ or 2 for x in $G(t,x)$.

CA(150) requires the most extensive analysis, and produces the most valuable new results, and accordingly is the sole topic of the second paper of the present set of three [3]. The claim made here depends on the following. For $\mathcal{N}$ for Rule 150, there is the well-known and elegant formula, Wolfram's formula, given in [1]. The formula provides an expression for each $N(t)$ which factorises in a way that reflects the binary structure for t, with factors determined by the Jacobsthal numbers of A071053 [4]. When the problem of Rule 150 is attacked via a general approach to the GF $G(x,t)$, one of the outcomes is the exposure of a factorisation structure for the $B(t)$ in exact correspondence with that of Wolfram's formula for $N(t)$. Further, the factors arising for the $B(t)$ are evaluated in [3].

The third paper [5] considers one simple two state two colour Turing Machine (TM), and the systematic occurrence in its evolution from a suitable initial state of the display lines of various CAs. The TM in question is the one labelled (f) on p79 of NKS [2], named as TM1953 using the numbering scheme for TMs employed in NKS. For each CA with an even numbered rule, the line numbers in the evolution of TM1953 where the CA's display lines are located, defines an integer sequence additional to the ones in focus here.

If the lines of any of the display for any CA $\in \Sigma$ are indexed by 'times' $t = 0, 1, \ldots,$ then the t-th line defines the integers

(i) $N(t)$, the (Hamming) weight of the line, which equals the number of ones it contains,
(ii) $B(t)$, the decimal integer of which the t-th display line is the binary representation,
(iii) $D(t)$, the decimal or denary integer represented by the display line itself.

The integer sequences to be treated here in a unified fashion are therefore the following three:

$$\mathcal{N} = \{N(t), \quad t = 0, 1, \ldots\}, \qquad \mathcal{B} = \{B(t), \quad t = 0, 1, \ldots\}, \qquad \mathcal{D} = \{D(t), \quad t = 0, 1, \ldots\}. \quad (4)$$

Usually displays for the rules of (2) and (3) are presented in centre-justified format. But here in displays presented below a left-justified format is employed. This is convenient because the paper deals with GFs, as defined in Definition 1 below, which contain only non-negative powers of x. It sometimes aids recognition of patterns in the displays.

Let $r = 0, 1, 2, \ldots,$ and $s = 0, 1, 2, \ldots,$ label the rows and columns of displays such as those in the Figures displayed below, and of any extended versions of them that need to be inspected.

Definition 1. The generating function of the display, from a single seed initial state, for each CA $\in \Sigma$ is given by

$$G(t,x) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} g_{rs} t^r x^s, \quad (5)$$

where $g_{rs} \in \{0,1\}$ is the entry at the intersection of row r and column s of the display.

Definition 2. The generating function of the sequence $\mathcal{N}$ the integers $N(t)$ is given by

$$G(t,1) = \sum_{r=0}^{\infty} N(r) t^r, \quad \text{where} \quad N(r) = \sum_{s=0}^{\infty} g_{rs}. \quad (6)$$

Definition 3. The generating function of the sequence of $\mathcal{B}$ of integers $B(t)$ is given by

$$G(t,2) = \sum_{r=0}^{\infty} B(r) t^r, \quad \text{where} \quad B(r) = \sum_{s=0}^{\infty} g_{rs} 2^s. \quad (7)$$

Definition 4. The generating function of the sequence of $\mathcal{D}$ of the decimal or denary integers $D(t)$ is given by

$$G(t,10) = \sum_{r=0}^{\infty} D(r)t^r, \quad \text{where} \quad D(r) = \sum_{s=0}^{\infty} g_{rs}10^s. \tag{8}$$

The last definition is included here because it allows $G_{60}(t,10)$, given here for CA(60), to be compared with a related but distinct result presented on p1091 of NKS [2].

Since 2^k occurs frequently here and in [3,5], it is convenient to employ the notation

$$\omega(k) = 2^k, \quad k = 0, 1, \ldots \; . \tag{9}$$

For background material on CAs, the seminal paper [1] should be consulted first. Then for a vast amount of profound and fascinating discussion of CAs, TMs, and much else, the book NKS [2] should be studied. The excellent index of NKS is a great aid to such study. The Wolfram Web resource MathWorld [7] also offers much detailed material information useful to work on CAs, TMs, etc. For background information on GFs read [8], or else refer to [9]. For a 'divide-and-conquer' type approach to GFs, often applied to contexts of interest here, especially in [5], see [6].

The material of the paper is organised as follows. In Section 2 displays for the CAs studied are presented in a series of Figures. In Section 3, there is a listing of the expressions derived for $G_R(t,x)$ for all CA(R) $\in \Sigma$, except for CA(150). Indication of the derivations of the expression obtained for $G_R(t,x)$ for R= 60, 22, 90, 220, 50, 54, $\in \Sigma_L$ is in Section 4. Section 5 deals with CA(94), and Section 6 with $G_R(t,x)$ for R= 28, 92, 158, 188, 190, $\in \Sigma_I$. Sections 7 − 9 are then devoted to the more difficult cases of $G_R(t,x)$ for R=62,182,126. Scattered throughout these sections are remarks about the special cases of $G_R(t,x)$ when $x = 1$ and $x = 2$, often without actually writing out the results, explicitly, and usually with a view to making contact with already known results. Section 10 discusses $\mathcal{D}$ and $G_R(t,10)$, focussing attention on CA(60), and bringing the use of modulo two arithmetic into view.

Many of the sequences $\mathcal{N}$ and $\mathcal{B}$ that obtain for the CAs studied here are well-known and well-studied, and feature, with much relevant information, including citation of their origin in cellular automaton theory, in that invaluble compilation, the On-Line Encyclopedia of Integer Sequences (OEIS) [4] founded in 1964 by N. J. A. Sloane, and enhanced systematically since then. For each of the CAs $\in \Sigma$, except CA(150), a display of their evolution from a single seed initial state is presented in the Figures below. The Figures all present displays in a left-justified format, which may be compared with the corresponding displays in centre-justified format in Fig. 3 of [1] and on pp 55,56 of NKS [2]. In association with the display for each CA(R) $\in \Sigma$, various items of information are collected in one place for ease of reference. In the display itself, early terms of $\mathcal{N}$ and $\mathcal{B}$ are shown in the last two columns. In the caption, the equation number for $G_R(x,t)$ here is provided, and the OEIS numbers for $\mathcal{N}$ and $\mathcal{B}$ for R are listed, whenever these are recorded [4] and non-trivial. Further, if results for $G_R(t,1)$ and/or $G_R(t,2)$, are already known, citation details are given too.

2 Displays for CAs $\in \Sigma$

The approach to getting formulas for GFs $G_R(x,t)$ often depends on either ideas related to the self-similarity structure of a CA display, or else to its block structure. The former is usually easily visible, but as an aid to exploiting the latter in the treatment of their GFs, a sixteen line display for the CAs of rules 62, 182, 126 is presented in a way designed to make it clear what the blocks are and where they are situated in the display. This is detailed in the captions. Fig. 10 for CA(94) shows that its display can be viewed as the superposition of two simpler ones.

2	0	1	1
222	1	3	7
22. . 3	2	3	19
11. 3333	3	6	123
11. 33. . . 2	4	5	283
11. 33. 4. 222	5	8	1883
11. 11. 4422. . 3	6	9	5083
11. 11. 44. . . 3333	7	10	30939
11. 11. 44. 4. 33. . . 2	8	10	72411
11. 11. 11. 4433. 5. 222	9	14	483035
11. 11. 11. 44. . . 5522. . 3	10	13	1296091
11. 11. 11. 44. 4. 55. . . 3333	11	15	7919323
11. 11. 11. 11. 4455. 5. 33. . . 2	12	16	18544347
11. 11. 11. 11. 44. . . 5533. 6. 222	13	18	123614939
11. 11. 11. 11. 44. 4. 55. . . 6622. . 3	14	18	331790043
11. 11. 11. 11. 11. 4455. 5. 66. . . 3333	15	21	2027402971

Fig. 1: Display of the evolution of CA(62)

All the entries $2, \ldots, 6$ stand for ones in the actual display. They serve to indicate which blocks occur and where they lie. The entries in the last two columns give t, $N(t)$, and $B(t)$ values of the lines of the display.

$G_{62}(t, x)$ is given by (47).

Here $\mathcal{N}$ is the sequence A071047 in OEIS, [4]. For $G_{62}(t, 1)$, see (48). This is a simpler version of a result given by H. Havermann in the entry for this $\mathcal{N}$ in [4]

$\mathcal{B}$ is not found in [4]. $G_{62}(t, 2)$ is not shown.

1	0	1	1
222	1	3	7
1. 2. 1	2	3	21
4444444	3	7	127
1. 44444. 1	4	7	381
222. 444. 222	5	9	1911
1. 2. 1. 4. 1. 2. 1	6	7	5461
888888888888888	7	15	32767
1. 8888888888888. 1	8	15	98301
222. 88888888888. 222	9	17	491511
1. 2. 1. 888888888. 1. 2. 1	10	15	1408981
4444444. 8888888. 4444444	11	21	8355711
1. 44444. 1. 88888. 1. 44444. 1	12	19	25001341
222. 444. 222. 888. 222. 444. 222	13	21	125269879
1. 2. 1. 4. 1. 2. 1. 8. 1. 2. 1. 4. 1. 2. 1	14	15	357913941
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	15	31	2147483647

Fig. 2: Display of the evolution of CA(182)

Here the numbers 1, 2, 4, 8 stand for the ones of the true display, which serves to indicate which n-blocks the these ones belong to. To economise on column width, in the last line x stands for 16.

$G_{182}(t, x)$ is given by (51) and (55).

$\mathcal{N}$ is sequence A071042 in [4]. For $G_{182}(t, 1)$, see Section 8.1.

$\mathcal{B}$ was not found in [4]. For $G_{182}(t, 2)$, see Section 8.2.

1	0	1	1
1 X1	1	3	7
1 1 . 1 1	2	4	27
1 1 1X111	3	7	127
1 1 . . . . . 1 1	4	4	387
1 1 11 . . . 1 111	5	8	1935
1 1 . . 11 . 1 1 . . 11	6	8	6579
1 1 11 111X1111111	7	15	32767
1 1 . . . . 1 1	8	4	98307
1 1 11 . . . . 1 111	9	8	491535
1 1 . . 11 . . . . 1 1 . . 11	10	8	1671219
1 1 11 1111 . . . . 1 1111111	11	16	8356095
1 1 . . . . . . 11 . . . . . 1 1 11	12	8	25264899
1 1 11 . . . . 1111 . . . 1 111 1111	13	16	126324495
1 1 . . 11 . . 11 . . 11 . 1 1 . . 11 . . 11 . . 11	14	16	429503283
1 1 11 1111 1111111X111111111111111	15	31	2147483647

Fig. 3: Display of the evolution of CA(126)

Each symbol X stands in the position of the one that the true display calls for. This serves to draw attention to those sites at which there is block overlap. Such sites may usefully be decreed to contribute twice – each one's one being called for to complete each of a pair of blocks. A subtraction term in $G_{126}(t,x)$ is therefore needed to compensate for this point of view.
$G_{126}(t,x)$ is given by (67).
$\mathcal{N}$ is the sequence A071051 in [4]. $G_{126}(t,1)$, (68) here, agrees with the result, (69) here, given by N. J. A. Sloane in the entry for this $\mathcal{N}$ in [4]
$\mathcal{B}$ was not found in [4]. For $G_{126}(t,2)$ see (75) here.

	t	$N(t)$	$B(t)$
1	0	1	1
1 1	1	2	3
1 . 1	2	2	5
1 1 1 1	3	4	15
1 . . . 1	4	2	17
1 1 . . 1 1	5	4	51
1 . 1 . 1 . 1	6	4	85
1 1 1 1 1 1 1 1	7	8	255
1 1	8	2	257
1 1 . . . . . . 1 1	9	4	771
1 . 1 . . . . . 1 1	10	4	1285
1 1 1 1 . . . . 1 1 1 1	11	8	3855
1 . . . 1 . . . 1 . . . 1	12	4	4369
1 1 . . 1 1 . . 1 1 . . 1 1	13	8	13107
1 . 1 . 1 . 1 . 1 . 1 . 1 . 1	14	8	21845
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	15	16	65535

Fig. 4: Display of and data for CA(60)

$G_{60}(t,x)$ is given by (10).
$\mathcal{N}$ is sequence A001316 in [4]. $G_{60}(t,1)$, (27) here, agrees with the result given by R. Stephan in the entry for this $\mathcal{N}$ in [4]. See also [10] for this and related results.
$\mathcal{B}$ is sequence A001317 in [4]. For $G_{60}(t,2)$ see (29) to (31) here. The form (29) agrees with the result given by S. Shakirov under the entry for this $\mathcal{B}$ in [4], where it is stated that it has been

proved by V. Shelelev. What is proved here is the general result (10).

	t	$N(t)$	$B(t)$
1	0	1	1
1 1	1	2	3
1 1 1	2	3	7
1 1 1 1	3	4	15
1 1 1 1 1	4	5	31
1 1 1 1 1 1	5	6	63
1 1 1 1 1 1 1	6	7	127
1 1 1 1 1 1 1 1	7	8	255

Fig. 5: Display of and data for CA(220)

$G_{220}(t, x)$ is given by (13).
$\mathcal{B}$ is sequence A000225 in [4]. The result of setting $x = 2$ in (13) agrees with the result given in [7] for CA(220), and simply quoted in [4].

	t	$N(t)$	$B(t)$
1	0	1	1
1 1 1	1	3	7
1 . . . 1	2	2	17
1 1 1 1 1 1	3	6	119
1 1	4	2	257
1 1 1 . . . . . 1 1 1	5	6	1799
1 . . 1 . . . 1 . . . 1	6	4	4369
1 1 1 . 1 1 1 . 1 1 1 . 1 1 1	7	12	30583

Fig. 6: Display of and data for CA(22)

$G_{22}(t, x)$ is given by (11).
$\mathcal{N}$ is sequence A071044 in [4]. $G_{22}(t, 1)$ here agrees with the result given by N. J. A. Sloane in the entry in [4] for this $\mathcal{N}$.
$\mathcal{B}$ for CA(22) is not found in [4]. For $G_{22}(t, 2)$, see (37).

	t	$N(t)$	$B(t)$
1	0	1	1
1 . 1	1	2	5
1 . 1 . 1	2	3	21
1 . 1 . 1 . 1 .	3	4	85
1 . 1 . 1 . 1 . 1	4	5	341
1 . 1 . 1 . 1 . 1 . 1	5	6	1365
1 . 1 . 1 . 1 . 1 . 1 . 1	6	7	5461
1 . 1 . 1 . 1 . 1 . 1 . 1 . 1	7	8	21845

Fig. 7: Display of and data for CA(50)

$G_{50}(t, x)$ is given by (14).
$\mathcal{B}$ is sequence A0002450 in [4].The result of setting $x = 2$ in (14) agrees with the result given by W. Lang in the entry in [4] for this $\mathcal{N}$, and also in [7] under CA(50).

	t	$N(t)$	$B(t)$
1	0	1	1
1 1 1	1	3	7
1 . . . 1	2	2	17
1 1 1 . 1 1 1	3	6	119
1 . . . 1 . . . 1	4	3	273
1 1 1 . 1 1 1 . 1 1 1	5	9	1911
1 . . . 1 . . . 1 . . . 1	6	4	4369
1 1 1 . 1 1 1 . 1 1 1 . 1 1 1	7	12	30583

Fig. 8: Display of and data for CA(54)

$G_{54}(t, x)$ is given by (15).
$\mathcal{N}$ is sequence A064455 in [4].The result of setting $x = 1$ in (15) agrees with the result given by R. Mathar in the entry in [4] for this $\mathcal{N}$.
$\mathcal{B}$ is sequence A118108 in [4]. The result of setting $x = 2$ in (15) agrees with the result given in [7] for CA(54).

	t	$N(t)$	$B(t)$
1	0	1	1
1 . 1	1	2	5
1 . . . 1	2	2	17
1 . 1 . 1 . 1	3	4	85
1 1	4	2	257
1 . 1 . . . . . 1 . 1	5	4	1285
1 . . . 1 . . . 1 . . . 1	6	4	4369
1 . 1 . 1 . 1 . 1 . 1 . 1 . 1	7	8	21845

Fig. 9: Display of and data for CA(90)

$G_{90}(t, x)$ is given by (12).
$\mathcal{N}$ for CA(90) is the same as $\mathcal{N}$ for CA(60).
$\mathcal{B}$ is sequence A038183 in [4]. For $G_{90}(t, 2)$, see (39)

	t	$N(t)$	$B(t)$
1	0	1	1
2 1 2	1	3	7
1 2 . 2 1	2	4	27
2 1 2 . 2 1 2	3	6	119
1 2 . 2 . 2 . 2 1	4	6	427
2 1 2 . 2 . 2 . 2 1 2	5	8	1879
1 2 . 2 . 2 . 2 . 2 . 2 1	6	8	6827
2 1 2 . 2 . 2 . 2 . 2 . 2 1 2	7	10	30039

Fig. 10: Display of and data for CA(94)

Writing a two for the one called for in the true display highlights the fact that the display is the superposition of two simpler ones, so that $G_{94}(t,x)$ is he sum of their separate contributions.
$G_{94}(t,x)$ is given by (17).
$\mathcal{N}$ is sequence A064880 in [4]. For $G_{94}(t,1)$ see (44) here.
$\mathcal{B}$ is sequence A118101 in [4]. The formula for $G_{94}(t,2)$, which (17) provides, is not quoted, but agrees with the result given in [7] for CA(94).

	t	$N(t)$	$B(t)$
1	0	1	1
1 1 1	1	3	7
1 1 1 1 1	2	5	31
1 1 1 1 1 1 1	3	7	127
1 1 1 1 1 1 1 1 1	4	9	511
1 1 1 1 1 1 1 1 1 1 1	5	11	2047
1 1 1 1 1 1 1 1 1 1 1 1 1	6	13	8191
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	7	15	32767

Fig. 11: Display of and data for CA(222)

$G_{222}(t,x)$ is given by (16), which, for $x=1$, generates the sequence A005408 [4] of odd integers.
$\mathcal{B}$ is sequence A083420 in [4]. The result of setting $x=2$ in (16) agrees with the result given in [7] for CA(222).

	t	$N(t)$	$B(t)$
1	0	1	1
1 1	1	2	3
1 . 1	2	2	5
1 . 1 1	3	3	13
1 . 1 . 1	4	3	21
1 . 1 . 1 1	5	4	53
1 . 1 . 1 . 1	6	4	85
1 . 1 . 1 . 1 1	7	5	213

Fig. 12: Display of and data for CA(28)

$G_{28}(t, x)$ is given by (18).
$\mathcal{B}$ is sequence A086893 in [4]. The result of setting $x = 2$ in (18) agrees with the result given by P. Barry in the entry for this $\mathcal{B}$ in [4].

	t	$N(t)$	$B(t)$
1	0	1	1
1 1	1	2	3
1 1 1	2	3	7
1 . 1 1	3	3	13
1 . 1 1 1	4	4	29
1 . 1 . 1 1	5	4	53
1 . 1 . 1 1 1	6	5	117
1 . 1 . 1 . 1 1	7	5	213

Fig. 13: Display of and data for CA(92)

$G_{92}(t, x)$ is given by (19).
$\mathcal{N}$ is sequence A206916 in [4].
$\mathcal{B}$ is sequence A230418 in [4].

	t	$N(t)$	$B(t)$
1	0	1	1
1 1 1	1	3	7
1 1 1 . 1	2	4	23
1 1 1 . . 1 1	3	5	103
1 1 1 . 1 1 1 . 1	4	7	375
1 1 1 . . 1 1 . . 1 1	5	7	1639
1 1 1 . 1 1 1 . 1 1 1 . 1	6	10	6007
1 1 1 . . 1 1 . . 1 1 . . 1 1	7	9	26215

Fig. 14: Display of and data for CA(158)

$G_{158}(t, x)$ is given by (20).
$\mathcal{N}$ is sequence A071054 in [4]. The result $G_{158}(x, 1)$ of setting $x = 1$ in (20) is given by agrees the result given, but unattributed, in the entry for this $\mathcal{N}$ in [4].
$\mathcal{B}$ is not found in [4]. The formula for $G_{158}(x, 2)$ is not quoted.

	t	$N(t)$	$B(t)$
1	0	1	1
1 1	1	2	3
1 . 1	2	2	5
1 1 1 1	3	4	15
1 1 1 . 1	4	4	23
1 1 . 1 1 1	5	5	59
1 . 1 1 1 . 1	6	5	93
1 1 1 1 . 1 1 1	7	7	239

Fig. 15: Display of and data for CA(188)

$G_{188}(t,x)$ is given by (21).

Neither $\mathcal{N}$ nor $\mathcal{B}$ is found in [4]. No formula is quoted either for $G_{198}(x,1)$ or for $G_{188}(x,2)$.

	t	$N(t)$	$B(t)$
1	0	1	1
1 1 1	1	3	7
1 1 1 . 1	2	4	23
1 1 1 . 1 1 1	3	6	119
1 1 1 . 1 1 1 . 1	4	7	375
1 1 1 . 1 1 1 . 1 1 1	5	9	1911
1 1 1 . 1 1 1 . 1 1 1 . 1	6	10	6007
1 1 1 . 1 1 1 . 1 1 1 . 1 1 1	7	12	30583

Fig. 16: Display of and data for CA(190)

$G_{190}(t,x)$ is given by (22).

$\mathcal{N}$ is sequence A032766 in [4]. The result of setting $x = 1$ in (22) is not quoted. It agrees the result given, but unattributed, in the entry for this $\mathcal{N}$ in [4].

$\mathcal{B}$ is not found in [4], nor is a result for $G_{158}(x,2)$ quoted.

3 A listing of GFs for CAs $\in \Sigma$

$$G_{60}(t,x) = \prod_{k=0}^{\infty} \left[1 + t^{\omega(k)} + (xt)^{\omega(k)}\right], \quad \omega(k) = 2^k, \quad k = 0,1,\ldots \tag{10}$$

$$G_{22}(t,x) = \left[1 + t(1 + x + x^2)\right] G_{60}(t^2, x^4). \tag{11}$$

$$G_{90}(t,x) = \prod_{k=0}^{\infty} \left[1 + t^{\omega(k)} + (x^2 t)^{\omega(k)}\right]. \tag{12}$$

$$G_{220}(t,x) = \frac{1}{(1-t)(1-xt)}. \tag{13}$$

$$G_{50}(t,x) = \frac{1}{(1-t)(1-x^2 t)}. \tag{14}$$

$$G_{54}(t,x) = \frac{1 + t(1 + x + x^2)}{(1-t^2)(1-x^4 t^2)}. \tag{15}$$

$$G_{222}(t,x) = \frac{1 + xt}{(1-t)(1-x^2 t)}. \tag{16}$$

$$G_{94}(t,x) = \frac{(1+xt)\left[1 + (1+x^2)t - x^4 t^4\right]}{(1-t^2)(1-x^4 t^2)}. \tag{17}$$

$$G_{28}(t,x) = \frac{1}{(1-t)(1-x^2 t^2)} + \frac{xt}{(1-x^2 t^2)}. \tag{18}$$

$$G_{92}(t,x) = G_{28}(t,x) + \frac{xt^2}{(1-x^2 t^2)}. \tag{19}$$

$$G_{158}(t,x) = \frac{(1+t)(1+xt+x^2t) - x^4t^3}{(1-t^2)(1-x^4t^2)}. \tag{20}$$

$$G_{188}(t,x) = \frac{1}{(1-t)(1-xt)} - \frac{xt^2(1+xt^3)}{(1-x^2t^2)(1-t^4)}. \tag{21}$$

$$G_{190}(t,x) = \frac{1+xt(1+x)}{(1-t)(1-x^4t^2)}. \tag{22}$$

For results for $G_R(t,x)$ for R=62, 182,126, see Sections 7, 8, 9, and for CA(150), see [3].

4 Generating functions for the easier CAs

4.1 CA(60)

The display for CA(60) in Fig. 4 has a well-known self-similarity structure. It maps onto itself on a scale larger by a factor of two in each dimension under the mapping

$$1 \mapsto \mathrm{J} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad 0 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{23}$$

so that

$$G_{60}(t,x) = (1+t+xt)G_{60}(t^2,x^2), \tag{24}$$

where the factor $(1+t+xt)$ describes the matrix J in GF terms. The solution of (24) is given as (10) in Section 3.

There are other ways to derive this result. The first two rows of the display in Fig. 4 contribute to $G_{60}(t,x)$ the expression

$$(1+t+xt) = K(t,x), \tag{25}$$

and the first four rows contribute

$$K(t,x)\left[1+t^2+(xt)^2\right] = K(t,x)K(t^2,x^2), \tag{26}$$

and so on leading to (10).

The GFs for the sequences $\mathcal{N}$ and $\mathcal{B}$ follow from (10). First,

$$G_{60}(t,1) = \prod_{k=0}^{\infty}\left[1+2t^{\omega(k)}\right], \tag{27}$$

yielding the data in the $N(t)$ column of Fig. 4. Second, writing $G_{60}(t,1) = g(t)$, inspection of $\mathcal{N}$ suggests that $g(t)$ satisfies

$$g(t) = (1+2t)(1+2t^2)g(t^4), \tag{28}$$

for which (27) provides the solution. Eq. (28) implies that $g(t)$ is a GF of divide-and-conquer type as described in [6]. Third

$$\begin{aligned}
G_{60}(t,2) &= \prod_{k=0}^{\infty}\left[1+(2^{\omega(k)}+1)t^{\omega(k)}\right], && (29)\\
&= \prod_{k=0}^{\infty}\left[1+F_k t^{\omega(k)}\right], && (30)\\
&= (1+F_0t)(1+F_1t^2)(1+F_2t^4)\dots \\
&= 1+F_0t+F_1t^2+F_0F_1t^4+F_2t^4++F_0F_2t^5++F_1F_2t^6++F_0F_1F_2t^7+\dots \\
&= 1+3t+5t^2+15t^3+17t^4+51t^5+85t^6+255t^7+\dots, && (31)
\end{aligned}$$

where the Fermat numbers, see A000215 in [4], are defined by

$$F_r = 2^{\omega(r)}+1, \quad \omega(r) = 2^r, \quad r = 0,1,\dots, \tag{32}$$

so that $F_0 = 3, \quad F_1 = 5, \quad F_2 = 17$, etc. For detailed discussion of Fermat numbers, see [11]. From Theorem 8.1 there, it is seen that, if $K(r)$ is the set of integers such that $r = \sum_{k\in K(r)} 2^k$, then the $B(r) = B_{60}(r)$ are given by

$$B(0) = 1, \quad B(r) = \prod_{k\in K(r)} F_k, \quad r = 1, 2, \dots\ , \tag{33}$$

proved also in [12]. This paper contains a display like the early part of Fig. 4, and correlates its lines with the sequence $\mathcal{B}$. In [13], the same facts are attributed, see p137, to W. Watkins, but without the indication of a source.

4.2 CA(22)

The display for CA(22) in Fig. 6 is obtained from the display for CA(60) by means of the substitutions

$$\mathrm{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{M}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

for the numbers one and zero. To deduce an expression for $G_{22}(t, x)$ from (10), x and t must be replaced by x^4 and t^2 because M_1 and M_0 are 2×4 matrices, and the result thereby obtained must be multiplied by the factor $\left[1 + t(1 + x + x^2)\right]$ which describes the matrix M_1. Hence

$$G_{22}(t, x) = \left[1 + t(1 + x + x^2)\right] \prod_{k=1}^{\infty} \left[1 + t^{\omega(k)} + (x^2 t)^{\omega(k)}\right], \tag{35}$$

in agreement with (11) in Section three. Also

$$G_{22}(t, 1) = (1 + 3t) \prod_{k=1}^{\infty} \left[1 + 2t^{\omega(k)}\right], \tag{36}$$

and

$$G_{22}(t, 2) = (1 + 7t) \prod_{k=1}^{\infty} \left[1 + F_{k+1} t^{\omega(k)}\right], \tag{37}$$

where the Fermat numbers, as defined in (32) appear, in agreement with the data in the N and B columns of Fig. 6.

4.3 CA(90)

A self-similarity argument like that given in Sction 4.1 for Rule 60, here leads to the result (12) in Section 3.

This result also emerges by the elementary method used in Section 4.1 of examining the first two, then the first four rows of Fig. 9, etc.

Otherwise, the display in Fig. 4 for CA(60) maps into an extended version of the display of Fig. 9 for CA(90) under

$$1 \mapsto m_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad 0 \mapsto m_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{38}$$

Hence, as in Section 4.1, passage from (24) for $G_{60}(t, x)$ to (12) is achieved merely by replacing x by x^2, since m_1 and m_0 are 1×2 matrices. There is no multiplying factor since m_1 is described by 1. This approach to $G_{90}(t, x)$ is in some sense a formal way of exploiting the fact that the rows of Fig. 9 are the same as the even rows of Fig. 4.

The sequence $\mathcal{N}$ for CA(90) coincides with that of CA(60), and

$$G_{90}(t, 2) = \prod_{k=0}^{\infty} \left[1 + F_{k+1} t^{\omega(k)}\right]. \tag{39}$$

4.4 CA(R) for R= 50, 54, 220, 222

It is easy, by inspection of the displays of Figs. 5 and 7, to derive the results (13) for $G_{220}(t,x)$, and (14) for $G_{50}(t,x)$. The display for CA(220) of Fig. 5 maps to that of Fig. 8 for CA(54) under the mapping (34), so that (15) for $G_{54}(t,x)$ follows directly.

Inspection of the display of Fig. 11 for CA(222) leads readily to the result (16) for $G_{222}(t,x)$.

However, there is another argument, typical of the ones already used, for deriving (16). If each one in the display of Fig. 7 for CA(50) is simply overwritten by the two-by-two unit matrix I, then the display of Fig. 11 for CA(222) arises, without increase in size by a factor of two in either the x or t direction. Then to get an expression for $G_{222}(t,x)$, it is necessary only to multiply $G_{50}(t,x)$ by the factor $(1+xt)$ which describes I. The results (14) and (16) agree with this. Put otherwise, this argument offers an interpretation of the factor $(1+xt)$ in (16) for $G_{222}(t,x)$.

5 CA(94)

The display of Fig. 10 for CA(94) shows that it is the superposition of two independent displays, which map, under

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto A, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mapsto \bullet \tag{40}$$

to the displays

$$\left| \begin{array}{cccccccc} A & & & & & & & \\ A & \bullet & A & & & & & \\ A & \bullet & \bullet & \bullet & A & & & \\ A & \bullet & \bullet & \bullet & \bullet & \bullet & A & \end{array} \right| , \quad \left| \begin{array}{cccccccc} . & & & & & & & \\ A & A & & & & & & \\ A & A & A & A & & & & \\ A & A & A & A & A & A & & \\ A & A & A & A & A & A & A & A \end{array} \right| \tag{41}$$

Here the top-left element of the matrix A in the first of these arrays is at site (0, 0) whereas that in the second one is at (1, 0). The latter is a dot that stands for a number zero. As the relative positioning of the A rows suggests, the top left elements are in even t lines in the left display, and in odd t lines on the right display in (41). The first display of (41) gives a contribution to $G_{94}(t,x)$ equal to

$$G_1(t,x) = \frac{1+xt}{1-t^2} + \frac{(1+xt)x^4t^2}{1-x^4t^2}, \tag{42}$$

where A yields the factor $(1+xt)$. The second display in (41) consists of two versions of a CA(50) type of display, the second one shifted with respect to the first by one block to the right. Hence there is a contribution to $G_{94}(t,x)$ equal to

$$G_2(t,x) = (1+xt)(1+x^2)t\frac{1}{(1-t^2)(1-x^4t^2)}. \tag{43}$$

Here the first factor is for A, the second is for the block shift, and the factor t arises because the display starts at $t=1$. Also x and t in (14) for $G_{50}(t,x)$ have been replaced by x^2 and t^2 since A is a 2×2 block. Simplifying (42) and (43) gives the final result (17) given in Section 3.

The reader may prefer to obtain this (17) from a frontal attack on the display of Fig. 10, while the writer hopes the reader will find the above approach instructive, as the action moves towards the more difficult cases.

Eq. (17) simplifies for $x=1$ to give

$$G_{94}(t,1) = \frac{1+2t-t^4}{(1-t)(1-t^2)}, \tag{44}$$

as can be seen in [7] for R=94.

6 CA(R) with R =28, 92,158,188 190

The cases of CA(28) and CA(92) require no comment. Derivation of (20) for $G_{158}(t,x)$ is left as an exercise in the use of methods applied to cases already treated.

The result (21) for $G_{188}(t,x)$ is given in a form that indicates its origin. If all the zeros in the display of Fig. 15 for CA(188) are replaced by ones, the display of Fig. 5 for CA(220) is formed. To compensate for this, appropriate subtraction terms are needed for the two sets of left-to-right diagonal sequences of zeros that have been filled in. Since the leading elements of the two sets are located in Fig. 15 at $(x,t) = (1,2)$ and $(2,5)$, the rest of (21) should be clear.

The structure of (22) for $G_{190}(t,x)$ suggests that there is a substitution based argument for deriving it like that used for CA(22) and $CA(50)$. If each entry one of the display of Fig. 5 for CA(220) is replaced by the 2×4 matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

the display of Fig. 16 arises directly. While it is left to the reader, to complete the argument for getting (22), it must be seen why the factor $(1-t)$ appears in the denominator rather than $(1-t^2)$. The reason is that the M blocks overlap without any increase in the size in the t direction of the display of Fig. 16 relative to Fig. 5.

7 CA(62)

The display of Fig. 1 has, as its caption explains, been constructed to allow identification of the series of blocks. For the blocks marked by twos and threee, the block functions are

$$\begin{aligned} E_2 &= 1 + t(1+x+x^2) + t^2(1+x) \\ E_3 &= x + t(1+x+x^2+x^3) + (t^2+t^3)(1+x), \end{aligned} \tag{45}$$

allowing their contributions to $G_{62}(t,x)$ to be evaluated. Each of the blocks of fours, fives, sixes, etc., have block function

$$E_0 = 1 + (t+t^2+t^3)(1+x). \tag{46}$$

The separate contributions to $G_{62}(t,x)$ from blocks marked four, blocks marked five, and so on, can be calculated, and combined, leading to the following result

$$G_{62}(t,x) = \frac{E_0 x^6 t^5}{(1-x^3t^3)(1-x^8t^4)} + \frac{E_2}{(1-x^8t^4)} + \frac{E_3 x^3 t^2}{(1-x^8t^4)} + \frac{(1+x)t^3}{(1-t)(1-x^3t^3)}. \tag{47}$$

When $x = 1$, (47) simplifies to

$$G_{62}(t,1) = \frac{1}{1-t^3} + \frac{t(2t^2+3)}{(1-t)(1-t^4)}. \tag{48}$$

Computer algebra confirms that (47) yields an expression for $G_{62}(t,2)$ that correctly reproduces the results for $B(t)$ shown in Fig. 1.

8 CA(182)

Define blocks by inspection of Fig. 2 for CA(182), which has been drawn up to expedite their identification and placement.

The 1-blocks, which consist of a single entry equal to one, sit at sites in Fig. 2, which agree exactly with the display of Fig. 4 for CA(60), but spread out so that they contribute to $G_{182}(t,x)$ the term

$$G_1(t,x) = a_1 G_{60}(t^2, x^4), \tag{49}$$

where $a_1 = 1$ is the factor describing a 1-block. Similarly for the $\omega(k)$-blocks for $k = 1, 2, 4, \ldots$. Fig. 2 uses the numbers $2, 4, 8, \ldots$ to locate and depict them. Thus the 2-blocks provide

$$G_2(t, x) = a_2 t G_{60}(t^4, x^8), \tag{50}$$

where a_2 describes a 2-block. Proceeding in this manner, it is found that

$$G_{182}(x, t) = \sum_{k=0}^{\infty} a_{\omega(k)}(t, x) t^{\omega(k)-1} G_{60}(t^{2\omega(k)}, x^{4\omega(k)}), \tag{51}$$

where, it may be recalled, $\omega(k) = 2^k, \quad k = 0, 1, \ldots$.

It remains to achieve a measure of control over the block factors a_r. Writing

$$\begin{gathered} a_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad , \quad b_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \\ a_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \quad , \quad b_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & b_1 \\ b_1 & b_1 \end{pmatrix} \\ a_{2\omega} = \begin{pmatrix} a_\omega & b_\omega \\ 0 & a_\omega \end{pmatrix} \quad , \quad b_{2\omega} = \begin{pmatrix} b_\omega & b_\omega \\ b_\omega & b_\omega \end{pmatrix}, \end{gathered} \tag{52}$$

it is seen how to make systematic construction of the $a_{\omega(k)}$ factors.

The insertion of a column of zeros on the left-hand edge of the a-blocks may look slightly strange, but it is called for to allow (52) to be presented consistently, and does no harm to the calculation of $G_{182}(t, x)$, provided that use is made of

$$a_1 = 1, \quad \text{not} \quad a_1 = x, \quad \text{but} \quad b_1 = (1 + x), \tag{53}$$

and

$$a_2 = (1 + tx^2)a_1 + xb_1 = 1 + x + (t + 1)x^2, \quad \text{etc.} \tag{54}$$

A closed form expression for $a_{\omega(k)}$ can be found by calculating the contribution to it from each of the rows of the corresponding block, viewed as a geometric progression, and rearranging. The resultant expression is

$$a_{\omega(k)} = \frac{1 - (tx^2)^{\omega(k)}}{(1 - x)(1 - tx^2)} - \frac{x^{2\omega(k)-1}(1 - t^{\omega(k)})}{(1 - x)(1 - t)}, \tag{55}$$

for $k = 0, 1, \ldots$.

8.1 $G_{182}(t, 1)$

Eq. (55) is not helpful when $x = 1$, but it is straightforward to prove that

$$\begin{aligned} a_2(t, 1) &= 3 + t \\ a_4(t, 1) &= 7 + 5t + 3t^2 + t^3 \\ a_{\omega(k)}(t, 1) &= \sum_{s=0}^{\omega(k)-1} (\omega(k + 1) - 2s - 1)t^s, \end{aligned} \tag{56}$$

and to find

$$G_{182}(t, 1) = 1 + 3t + 3t^2 + 7t^3 + 7t^4 + 9t^5 + 7t^6 + 15t^7, \tag{57}$$

and the rest of the entries in the $N(t)$ column of Fig. 2. The sum in (56) can be evaluated in closed form.

8.2 $G_{182}(t, 2)$

For $x = 2$

$$a_1 = 1, \quad b_1 = 3, \quad a_2 = 7 + 4t, \quad b_2 = 15(1 + t), \tag{58}$$

and

$$\begin{aligned} a_4 &= 127 + 124t + 112t^2 + 64t^3 \\ b_4 &= 255(1 + t + t^2 + t^3) \\ a_8 &= (1 + 256t^4)a_4 + 128b_4, \end{aligned} \tag{59}$$

can be calculated in turn, or else (55) can be used for the $a_{\omega(k)}$. The entries in the $B(t)$ column of Fig. 2 can be confirmed as correct.

9 CA(126)

Inspection of Fig. 3 indicate that the first two lines need to be treated separately from the rest of the display. Their contribution to $G_{126}(t, x)$ will be written, to tidy up the final result, in the form

$$G_1(t, x) = 2 + t(1 + x)^2 - (1 + tx). \tag{60}$$

The remainder of the display exhibits a block structure complicated systematically, but only to a modest extent, by overlaps at sites with (x, t) values $(3, 3), (7, 7), (15, 15), \ldots$, marked with an X in Fig. 3, to highlight the issue.

In the $t = 2$ and 3 lines, there occur the $(2, 4)$-blocks

$$\mathrm{M}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{61}$$

with their top-leftmost elements at sites with (x, t) values (2,0) and (2,3), with the noted overlap at (3,3). The true display of course places a one at (3,3), while the need to deal with complete blocks of type M_2 calls for a two. To deal with the matter, act as if there is indeed a two at site (3,3), and compensate for this by explicitly subtracting the single term $(xt)^3$.

The block M_2 is described by the expression

$$\mu_2 = (1 + x)K(t, x^2), \tag{62}$$

where $K(t, x)$ is defined by (25), and the contribution from M_2-blocks is

$$G_2(t, x) = t^2(1 + x)K(t, x^2)(1 + x^3) - (xt)^3. \tag{63}$$

The approach works in exactly the same way for the rest of the display of Fig. 3, with *e.g.* $(4, 8)$-blocks

$$\mathrm{M}_4 = \begin{pmatrix} \mathrm{M}_2 & 0 \\ \mathrm{M}_2 & \mathrm{M}_2 \end{pmatrix}, \tag{64}$$

with an overlap at $(x, t) = (7, 7)$ and top-leftmost elements at $(x, t) = (4, 0)$ and $(4, 7)$. Also the factor describing M_4 can be written as

$$\mu_4 = \mu_2(1 + t^2 + x^4t^2) = (1 + x)K(t, x^2)K(t^2, x^4), \tag{65}$$

so that

$$G_2(t, x) = t^4(1 + x)K(t, x^2)K(t^2, x^4)(1 + x^7) - (xt)^7, \tag{66}$$

and finally

$$\begin{aligned} G_{126}(t, x) &= 2 + t(1 + x)^2 - \sum_{r=0}^{\infty}(xt)^{\omega(r)-1} \\ &\quad + (1 + x)\sum_{r=1}^{\infty} t^{\omega(r)} \left[1 + x^{2\omega(r)-1}\right] \prod_{k=0}^{r-1} K(t^{\omega(k)}, x^{2\omega(k)}). \end{aligned} \tag{67}$$

9.1 $G_{126}(t,1)$

When $x=1$, (67) gives

$$G_{126}(t,1) = 2(1+2t) - \sum_{r=0}^{\infty} t^{\omega(r)-1} + 4\sum_{r=1}^{\infty} t^{\omega(r)} \prod_{k=0}^{r-1}\left[1+2t^{\omega(k)}\right]. \tag{68}$$

However a simpler result is given [4] under the entry for A071051, as noted in the caption to Fig. 3:

$$G_{126}(t,1) = 2\prod_{k=0}^{\infty}\left[1+2t^{\omega(k)}\right] - \sum_{r=0}^{\infty} t^{\omega(r)-1}. \tag{69}$$

To see that the two results agree, use must be made of the identity

$$2\sum_{r=1}^{\infty} t^{\omega(r)} \prod_{k=0}^{r-1}\left[1+2t^{\omega(k)}\right] + 1 + 2t = \prod_{k=0}^{\infty}\left[1+2t^{\omega(k)}\right]. \tag{70}$$

Further, (69) draws attention to the identity between sequences

$$A071051 = 2.A001316 - A255738, \tag{71}$$

where [4]

$$A255738 = 11010^3 10^7 10^{15} 1\ldots, \tag{72}$$

related evidently to the subtraction term necessitated by the overlap issue.

9.2 $G_{126}(t,2)$

In the second line of (67), there occur the factor $(1+x)$ and the product factor. When $x=2$, these feature in the identity

$$F_0 \prod_{k=0}^{r-1}\left[1+F_{k+1}t^{\omega(k)}\right] = \sum_{j=0}^{\omega(r)-1} B(2j+1)t^j, \tag{73}$$

where the Fermat numbers are defined by (32), and the numbers $B(r)$ in use in (73), $B(r) = B_{60}(r)$, are defined by (33). Hence

$$\begin{aligned} G_{126}(t,2) &= 2+9t-\sum_{r=0}^{\infty}(2t)^{\omega(r)-1} \\ &\quad + \sum_{r=1}^{\infty} t^{\omega(r)}\left[1+2^{2\omega(r)-1}\right]\sum_{j=0}^{\omega(r)-1} B(2j+1)t^j, \end{aligned} \tag{74}$$

which is easy to deal with, giving

$$\begin{aligned} G_{126}(t,2) &= \sum_{r=0}^{15} B_{126}(r)t^r \\ &= 2+9t-(1+2t+8t^3+128t^7+32768t^7) \qquad (75)\\ &+ 9t^2\left[B(1)+B(3)t\right] + 129t^4\left[B(1)+\ldots B(7)t^3\right] + 32769t^8\left[B(1)+\ldots B(15)t^7\right]+\ldots\quad . \end{aligned}$$

Eq. (74) makes clear the way the $B_{126}(r)$ factorise in terms of the Fermat numbers, requiring corrections only for $t^{2\omega(r)-1}$ terms.

10 $G(t,x)$ for $x = 10$

According to Definition 4, $G(t,10)$ generates via the coefficients of its t^r term, the r-th display line in the display of the evolution of a CA from its single seed initial state. This means that it produces this coefficient as an ordinary decimal or denary integer. It is enough to discuss only the case of CA(60). Eq. (10) for $G_{60}(t,x)$ gives rise to

$$\begin{aligned} G_{60}(t,10) &= \prod_{k=0}^{\infty} \left[1 + t^{\omega(k)}(1 + 10^{\omega(k)})\right] \qquad (76) \\ &= [1 + 11t]\left[1 + 101t^2\right]\left[1 + 10001t^4\right]\ldots \\ &= 1 + 11t + 101t^2 + 1111t^3 + 10001t^4 + 1100011t^5 + 1010101t^6 + 1111111t^7\ldots, \end{aligned}$$

where ordinary multiplication is used *e.g.* for the t^3 and t^6 terms: $11 \times 101 = 1111$ and $101 \times 10001 = 1010101$. The lines of Fig. 4 have made their expected appearance. No use of modulo two arithmetic has been made. Indeed the issue has not arisen at all. The same statements apply to all CAs $\in \Sigma$.

A different type of formula for the generating functions of the display lines of CAs including CA(60) can be found in NKS [2], p1091. There the formula

$$\begin{aligned} g_{60}(t,x) &= [1 - t(1 + x)]^{-1} \qquad (77) \\ &= \sum_{n=0}^{\infty} t^n(1 + x)^n. \end{aligned}$$

is given. As is very well-known, this produces the results seen in (76), but in this case upon the use of modulo two arithmetic.

To understand the situation further, use of the well-known identity

$$(1 - y)^{-1} = \prod_{n=0}^{\infty} \left[1 + y^{\omega(n)}\right], \qquad (78)$$

whose n-th term provides the binary representation of n, translates (77) into the alternative form

$$g_{60}(t,x) = \prod_{n=0}^{\infty} \left[1 + t^{\omega(n)}(1 + x)^{\omega(n)}\right], \qquad (79)$$

to be compared with (10), written now in the form

$$G_{60}(t,x) = \prod_{n=0}^{\infty} \left[1 + t^{\omega(n)}(1 + x^{\omega(n)})\right]. \qquad (80)$$

It is obvious that the formulas for $g_{60}(t,x)$ and $G_{60}(t,x)$ are completely distinct results, each one valid in its own context.

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