Gauge/Gravity Duality
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Mathematical Tripos, Part III
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Gauge/Gravity Examples Sheet #1

1. There are a variety of different concepts of "weight" in the literature and it can be confusing to keep track of all of them. To help sort things out in your mind, consider a dilation in a scale-invariant QFT in d-dimensional Minkowski spacetime. The dilation is parameterised by a positive real number Ω. This symmetry transformation consists of performing the following three transformations:

   Coordinates A uniform rescaling of the d coordinates: $\mathbf{x}' = \Omega\mathbf{x}$,
   (also causes tensors to transform according to their index structure)

   Metric A constant Weyl transformation of the metric: $g'_{ab} = \Omega^2 g_{ab}$

   Fields A further rescaling of any weighted fields e.g. $\phi' = \Omega^{-\Delta_\phi}\phi$.

   Under a dilation, a quantity Q will transform as

$$Q' = \Omega^{-\Delta}Q; \qquad \Delta = w_C + w_M + w_F \tag{1}$$

   where the three weights w_C, w_M, w_F represent the power of the transformations under the coordinate, metric, and field transformations respectively.

   Consider now the following free field actions:

$$I_{\text{scalar}} = -\int d^d x \sqrt{-g}\, \partial_a \varphi^* \, \partial^a \varphi; \tag{2}$$

$$I_{\text{Weyl}} = i\int d^d x \sqrt{-g}\, \psi^\dagger \gamma^a \partial_a \psi; \tag{3}$$

$$I_{\text{Maxwell}} = -\frac{1}{4}\int d^d x \sqrt{-g} F_{ab} F^{ab}, \tag{4}$$

   where $F_{ab} = \partial_a A_b - \partial_b A_a$.

   (Note that the determinant of the metric is defined as

$$g = g_{ab} g_{cd} g_{ef\ldots} \epsilon^{ace\ldots} \epsilon^{bdf\ldots} \tag{5}$$

   where $\epsilon^{abc\ldots}$ is the permutation symbol with d indices. Like the Kronecker delta δ^a_b, it is defined to take the same value in every coordinate system. Also recall that $\{\gamma^a, \gamma^b\} = 2g^{ab}$.)

   Complete the following table:

Q	w_C	w_M	w_F	Δ
g_{ab}				
δ^a_b				
g^{ab}				
$\epsilon^{abc\ldots}$				
$\sqrt{-g}$				
γ^a				
∂_a				
$d^d x$				
$\int$				
φ				
ψ				
A_a				
$J^a_{(\phi)}$				
$J^a_{(\psi)}$				
$T^{ab}_{(\phi)}$				
$T^{ab}_{(\psi)}$				
$T^{ab}_{(A)}$				

where J^a is the conserved U(1) Noether current associated with multiplying ϕ or ψ by a phase $e^{i\alpha}$, while T^{ab} is the conserved stress-energy tensor derived from any of the three actions.

You may assume that: (a) all weights are additive under multiplication, (b) the action I is dilation invariant ($\Delta_I = 0$), and (c) in the case a quantity $Q(\mathbf{x})$ that is a function of position, by convention the weights are assigned based on how $Q(\mathbf{0})$ transforms.

2. Consider now a general Weyl transformation, in which $\Omega(\mathbf{x})$ is a arbitrary function of position, and there is also a rescaling of fields according to their field weight w_F.

   (a) Show that, although the actions (2) and (4) in problem #1 are scale-invariant, they are *not* invariant under Weyl transformations unless $d = 2$ (scalar) or $d = 4$ (Maxwell).

   (b) Suppose you have a scalar primary operator $\mathcal{O}$ with weight Δ. How does the derivative $\partial_a \mathcal{O}$ transform under the Weyl rescaling? Is $\partial_a \mathcal{O}$ primary under special conformal transformations?

(c) Show that the non-minimally coupled scalar action

$$I = -\frac{1}{2}\int d^d x\,\sqrt{-g}\,\left(\partial_a\varphi\,\partial^a\varphi + \xi R\varphi^2\right) \tag{6}$$

is Weyl invariant (up to total derivative terms), for a particular choice of ξ ("conformal couping") which depends on d. [Hint: use an infinitesimal Weyl rescaling $\omega = \delta \ln\Omega$ to keep the calculation cleaner.]

* 3. In Euclidean field theory, the adjoint operation † also reverses the direction of the imaginary time coordinate, e.g. if we Wick rotate in the usual way from Minkowski spacetime, $\tau^\dagger = (it)^\dagger = -it = -\tau$ since $i^\dagger = -i$. A peculiar consequence of this, is that the definition of the adjoint actually depends on the choice of foliation; in particular it is not the same in radial quantization as in temporal quantization!

(a) Find the adjoints of each of the generators of the conformal group in *radial quantization*, where the Hilbert space lives on slices of constant radius r from the origin, and the analogue of "time translation" is given by the dilation operator $D = -i(x\cdot\partial)$. [You may assume that all generators of the conformal group are self-adjoint in Lorentzian signature when acting on the Einstein universe $S_{\mathrm{d}-1}\times\mathbb{R}$.]

Is the spectrum of D real or imaginary? What does this imply about the values of Δ for the corresponding operators (under the state-operator map)?

(b) In a CFT in $d > 2$ dimensions, a primary field is defined as one that transforms trivially under special conformal transformations. This implies that if $|p\rangle$ is the corresponding prmary state on the cylinder $S_{d-1}\times R$, then

$$K_a|p\rangle = 0. \tag{7}$$

The n-th order 'descendents' of the primary are defined by acting on the primary with n momentum operators:

$$P_a P_b \ldots |p\rangle \tag{8}$$

Show that the structure of a unitary irrep of the conformal symmetry group is fully determined by the dimension Δ and $SO(d)$ spin of the primary operator.

(c) By evaluating the norm of the $n = 1$ descendents of a primary operator, derive the following unitary bounds:

$$\Delta_\phi \geq 0 \qquad \text{(scalar)} \tag{9}$$

$$\Delta_{V^a} \geq d-1 \qquad \text{(vector)} \tag{10}$$

$$\Delta_{S^{ab}} \geq d \qquad \text{(sym. traceless tensor)} \tag{11}$$

What is the significance of the cases where the equality is satisfied?

(d) By evaluating the norm of the $n = 2$ descendent $P_a P^a|\phi\rangle$ (a.k.a. the Laplacian), show that any scalar (besides the identity) must satisfy the stronger unitarity bound:

$$\Delta_\phi \geq \tfrac{d-2}{2} \qquad \text{(scalar)}. \tag{12}$$

* 4. The vacuum 2 point function of a scalar operator ϕ of dimension $\Delta > 0$ in d-dimensional Minkowski spacetime is

$$\langle\, \phi(\mathbf{y})\phi(\mathbf{x})\rangle = \lim_{\epsilon\to 0} \frac{C}{|\mathbf{x}-\mathbf{y}+i\epsilon\hat{t}|^{2\Delta}}, \tag{13}$$

where $\hat{t}$ is a unit time vector in the usual Minkowski coordinates $\mathbf{x} = (t, \vec{x})$,

(a) Assuming a free field in $d = 3+1$ spacetime dimensions (so $\Delta = 1$), show that the imaginary part of this expression (which is the commutator) is proportional (when $t > 0$) to the usual retarded Green's function for the 3+1 Laplace operator:

$$\mathrm{Im}\,\langle\phi(\mathbf{x})\phi(\mathbf{0})\rangle = \left\langle \tfrac{1}{2}[\phi(\mathbf{x}), \phi(\mathbf{0})]\right\rangle \propto \frac{1}{4\pi t}\delta(t - |\vec{x}|), \tag{14}$$

where we have used translation symmetry to send one of the operators to the origin.

(b) Now consider the case of general d and a general scalar operator with $\Delta = \frac{d-2}{2} + \eta$. When does the commutator $[\phi(\mathbf{y}), \phi(\mathbf{x})]$ vanish at timelike separation?

(c) For what values of η is the following *spatially* smeared expectation value finite (and hence, a well-defined operator acting on the CFT Hilbert space):

$$\left\langle \left(\int d^{d-1}\vec{x}\, f(\vec{x})\phi(t=0,\vec{x}) \right)^2 \right\rangle = \text{finite?} \tag{15}$$

where f is a smooth test function that falls off faster than any power at large spatial values (e.g. a Gaussian).

Hint: Do a spacetime Fourier transform, a.k.a. a spectral decomposition. Note that there is no time ordering symbol in this expression so the momentum space result will differ from the Feynman propagator, instead (as discussed in the lectures) it is given by

$$\langle 0|\phi(-q)\phi(p)|0\rangle \propto \delta^d(p-q)\theta(E-|\vec{p}|) \begin{cases} (-p^2)^{\Delta - d/2} & \Delta > \frac{d-2}{2} \\ \delta(p^2) & \Delta = \frac{d-2}{2} \end{cases} \tag{16}$$

(d) What happens if we smear in time instead?

$$\left\langle \left(\int dt f(t)\phi(t, \vec{x} = 0) \right)^2 \right\rangle = \text{finite?} \tag{17}$$

5. Consider the following weight d tensor operator in a free massless Klein-Gordon scalar field in $d > 2$ dimensions:

$$X_{ab} = \alpha\, \partial_a \varphi \partial_b \varphi + \beta\, \varphi \partial_a \partial_b \varphi + \gamma\, \eta_{ab} \partial_c \varphi \partial^c \varphi. \tag{18}$$

Using the fact that the 2 point function vanishes between two primary operators of different Δ (and therefore also between the corresponding descendants), determine which linear combinations of α, β, γ are primary, and which is a descendant of a primary.

Verify that the traceless primary tensor is conserved, and therefore functions as a stress-tensor T_{ab}, even though it differs from the usual Klein-Gordon stress tensor. Looking at (6), what do you think is the cause of this difference?

6. Using conformal symmetry, show that the 3 point function of three different scalar primary operators $\phi_{1,2,3}$, with dimensions $\Delta_{1,2,3}$ respectively, is given in Euclidean signature by:

$$\langle\, \phi_1(\mathbf{z})\phi_2(\mathbf{y})\phi_3(\mathbf{x})\,\rangle = \frac{C_{123}}{|\mathbf{x}-\mathbf{y}|^{\Delta_1+\Delta_2-\Delta_3}|\mathbf{x}-\mathbf{z}|^{\Delta_1+\Delta_3-\Delta_2}|\mathbf{y}-\mathbf{z}|^{\Delta_2+\Delta_3-\Delta_1}} \tag{19}$$

where C_{123} is a constant which you need not determine (which is related to the 'OPE coefficients' which depend on the specific CFT).

[Hint: first check that the above expression is invariant under Poincaré generators, dilations, and inversions. Can you use these to get any 3 points into specified positions?]

7. Check that the following metrics each describe a D-dimensional anti-de Sitter spacetime with unit radius (or a subset or quotient thereof). Identify coordinate transformations which connect them to each other.

$$ds^2 = \tilde{\eta}_{AB}\, dX^A dX^B, \qquad \text{w/ constraint: } \tilde{\eta}_{AB}\, X^A X^B = -1; \tag{20}$$

$$ds^2 = d\rho^2 - \cosh(\rho)^2 dt^2 + \sinh(\rho)^2 d\Omega_{D-2}^2; \tag{21}$$

$$ds^2 = \frac{1}{\cos(\theta)^2}\left[-dt^2 + d\theta^2 + \sin(\theta)^2 d\Omega_{D-2}^2\right]; \tag{22}$$

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \eta_{ij}\, dx^i\, dx^j\right]; \tag{23}$$

$$ds^2 = -dT^2 + \cos(T)^2\left[dr^2 + \sinh(r)^2 d\Omega_{D-2}^2\right]; \tag{24}$$

where η_{ij} is the $D-1$ dimensional Minkowski metric, and $\tilde{\eta}_{AB}$ is a flat metric in a space-time-time with $D+1$ total dimensions.

[Note that (24) looks like a recollapsing FLRW cosmology with $k=-1$ (negative curvature), but the apparent 'Big Bang' and 'Big Crunch' singularities are actually just coordinate artifacts. You may find it illuminating to sketch the patch of AdS which this geometry covers on the 'tin can' diagram.]

8. Identify the $D(D+1)/2$ Killing vectors of AdS which satisfy Killing's equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{25}$$

by writing them down explicitly in AdS-Poincaré coordinates (23).

Show that on the conformal boundary ($z=0$), these Killing vectors reduce to the $(d+1)(d+2)/2$ conformal Killing vectors on Minkowski spacetime.

9. I stand at a point p in the the middle of AdS and shoot a lightray out to the boundary. When the lightray hits the conformal boundary, is its integrated affine parameter finite or infinite? Explain your answer.

Please email me at aw846@cam.ac.uk if you find any errors.

If you wish to sign up for the online examples class and have not yet done so, please email Gonçalo Araujo-Regado (ga365@cam.ac.uk) as soon as possible, with your CRSid, student status (part III, PhD, etc.), and personal timezone. Problems with a star (*) will be marked.