Gauge/Gravity Duality
Aron C. Wall (aw846@cam.ac.uk)

Mathematical Tripos, Part III
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Gauge/Gravity Examples Sheet #2

* 1. A 1+1 dimensional Majorana spinor has two real-valued components, ψ_L and ψ_R, and has the following action in Minkowski spacetime:

$$\int d^2x \left(\tfrac{1}{2}\psi_L\partial_v\psi_L + \tfrac{1}{2}\psi_R\partial_u\psi_R + m\psi_L\psi_R\right) \tag{1}$$

where $\partial_v = \partial_t - \partial_z$ and $\partial_u = \partial_t + \partial_z$. For the special case $m = 0$, this action is conformally invariant (and breaks up into two noninteracting Majorana-Weyl spinors).

(a) How should you promote m to be position dependent in order to describe the behavior of a Majorana fermion in a Poincare-AdS spacetime?

[Note: properly coupling spinors to curved spacetime is somewhat tricky and requires e.g. the vielbein formalism, but the question has been worded in a way where you don't need to worry about this.]

(b) Identify the two possible boundary conditions at the CFT boundary $z = 0$, and derive the following relationship between the mass m and the dimension Δ of the corresponding boundary operator:

$$m = \left|\Delta - \tfrac{1}{2}\right|, \tag{2}$$

where $\Delta \geq 0$ for a spinor operator in a unitary theory.

[For purposes of solving this problem, you do not need to determine whether the concept of a $d = 1$ CFT actually makes sense. But recently there has been a lot of interest in a statistical model in $d = 1$ called the SYK model, which enjoys *approximate* conformal invariance and seems to be holographically dual to a geometry that is approximately AdS_2.]

2. Suppose you have a free quantum Maxwell field in AdS_6 spacetime. (Assume the boundary conditions are chosen so that the theory is unitary and conformal.) What is the energy level and degeneracy of the lowest energy (gauge-invariant) single photon state, i.e. the first excited state, as measured by a clock in the centre of AdS?

Can you write down a formula for which energies have at least one state? What does this particular spectrum imply about the time evolution of the system?

[Hint: don't try to solve the wave equation in AdS, instead find a way to use the CFT dual.]

* 3. General relativity in 3 spacetime dimensions has the property that there is no Weyl curvature. Hence the curvature of a pure vacuum solution is fixed by the Einstein equation:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = g_{ab} \Lambda \tag{3}$$

For $\Lambda < 0$, this implies that in the neighborhood of any point, a purely vacuum solution always looks exactly like AdS spacetime. (A similar statement holds for $\Lambda = 0$ or $\Lambda > 0$, but replacing "AdS" with Minkowski or dS respectively.) However, the solution may have a different topology from AdS.

(a) Suppose you have a CFT in 1+1 dimensions, which in a large N limit is holographically dual to classical GR (at low energies). Calculate the thermal partition function $Z(\beta, L)$ of the CFT on a cylinder of circumference L and with inverse temperature β. [Hint: do a Wick rotation to a torus of lengths L and β.]

In solving this problem you may determine the action of empty AdS by reference to the Casimir energy of a CFT on a cylinder:

$$E = -\frac{2\pi c}{12L} \tag{4}$$

where c is the central charge of the CFT, which is related to the bulk Newton's constant by $c = 3R_{\mathrm{AdS}}/2G$.

(b) Verify that your proposed formula obeys the Cardy duality formula which comes from exchanging the roles played by space and imaginary time:

$$Z(\beta, L) = Z(L, \beta) \tag{5}$$

(c) Identify the temperature T_{c} at which the CFT has a first order phase transition. Plot the (leading order) entropy $S(T)$ across the phase transition. Check that at high temperatures the entropy scales as $S \propto LT$, similar to a Boltzmann gas in 1+1 dimensions.

(d) The solution above the critical temperature should look like a Euclidean black hole metric. Analytically continue your metric back to Lorentzian signature and verify the existence of a horizon. (This is the metric of a *BTZ black hole* in 2+1 dimensions.) Check that $S(T)$ is proportional to the area of the black hole horizon.

4. Holographic entanglement entropy in 1+1 dimensions:

(a) Using the same CFT as in the previous problem, but now in the 1+1 Minkowski vacuum, calculate the holographic entanglement entropy of an interval of length L on the $t = 0$ Cauchy slice. [Hint: how should the holographic entropy surface behave under an inversion transformation?]

Calculate the holographic entanglement entropy of a region R consisting of the union of two intervals defined by the spacetime endpoints $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{x}_3, \mathbf{x}_4)$, where the two intervals are each required to be spacelike and also spacelike separated to each other, but the 4 points are *not* required all lie on the t = 0 slice. Verify that the entropy you calculate diverges in the limit that the outer endpoints of the intervals approach one another, or in the limit that one interval becomes tiny.

(b) Some 1+1 dimensional CFTs are "chiral", meaning that the Hilbert space decomposes into a left-moving sector and a right-moving sector:

$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R, \tag{6}$$

where excitations in the left (right) sector move left (right) at the speed of light, and the two sectors don't interact with each other. (A simple example would be a free Dirac fermion, which decomposes into a left moving Weyl fermion and a right moving Weyl fermion, but there are more elaborate examples such as the so-called "Monster CFT".)

Based on your results above, show that the holographic CFT is *not* chiral.

Please email me at aw846@cam.ac.uk if you find any errors.

If you wish to sign up for the online examples class, please email Gonçalo Araujo-Regado (ga365@cam.ac.uk) as soon as possible. Problems with a star (*) will be marked.