Questions and corrections are welcome: write to Berry Groisman on bg268@

The Two-Body problem

Consider two particles with masses m_1 and m_2 interacting through central force. Their Lagrangian is

$$L = \frac{m_1 \dot{\boldsymbol{r}}_1^2}{2} + \frac{m_2 \dot{\boldsymbol{r}}_2^2}{2} - V(r), \tag{1}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vectors of the particles with respect to a chosen reference frame (assume fixed origin), and

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{2}$$

is the relative position vector.

Let us split the total kinetic energy into the term associated with motion of the centre of mass (CoM) and the term associated with the motion of the particles relative to each other.

Let us work through the algebra. Start with substituting for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from (2) into the definition of the CoM, $\boldsymbol{R}$, to obtain

$$\boldsymbol{R} = \boldsymbol{r}_1 - \frac{m_2}{m_1 + m_2} \boldsymbol{r} = \boldsymbol{r}_2 + \frac{m_1}{m_1 + m_2} \boldsymbol{r}. \tag{3}$$

Thus,

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{R}} + \frac{m_2}{m_1 + m_2} \dot{\boldsymbol{r}}, \quad \dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{R}} - \frac{m_1}{m_1 + m_2} \dot{\boldsymbol{r}}. \tag{4}$$

Hence, L can be rewritten as

$$L = \frac{1}{2}(m_1 + m_2)\dot{\boldsymbol{R}}^2 + \frac{\mu \dot{\boldsymbol{r}}^2}{2} - V(r), \tag{5}$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the *reduced mass*.

Now consider symmetries. First, *homogeneity of space* implies conservation of the total linear momentum, that is $\dot{\boldsymbol{R}}$ is constant - the CoM moves with constant velocity. Of course, we can deduce this by observing that $\boldsymbol{R}$ is ignorable/cyclic coordinate in the Lagrangian. (In one of the example questions in 1A Dynamics and Relativity you have shown this from the second Newton's law).

Let us work in the reference frame of the CoM. We can write

$$L = \frac{\mu \dot{\boldsymbol{r}}^2}{2} - V(r). \tag{6}$$

Thus, is we effectively reduced the 2-body problem to a 1-body problem.

Second symmetry, *isotropy of space*, implies that the vector of the angular momentum (with respect to the CoM), $\boldsymbol{L}_{CoM} = \boldsymbol{r} \times \boldsymbol{p}$, is conserved. Here $\boldsymbol{p}$ is the momentum conjugate to $\boldsymbol{r}$, which can be calculated in a standard way

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = \mu \dot{\boldsymbol{r}}. \tag{7}$$

Notice that this formally looks as if we are dealing with a single object of mass μ. [1]

We proceed with a standard argument (which had been used quite often in 1A Dynamics&Relativity) that since $\boldsymbol{L}$ is perpendicular to $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ the motion is confined to the plane perpendicular to $\boldsymbol{L}$. We now can introduce polar coordinates (r, ϕ) in this plane and rewrite

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r). \tag{8}$$

We analyse this form of the Lagrangian in a familiar way. First, observe that ϕ is ignorable, hence

$$\frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = l, \tag{9}$$

where $l \equiv |\boldsymbol{L}|$, which simply confirms conservation of the magnitude of the angular momentum. Euler-Lagrange equation for r gives

$$\mu\ddot{r} - \mu r \dot{\phi}^2 + \frac{\partial V}{\partial r} = 0. \tag{10}$$

We introduced the *effective potential* and rewrite the above expression as

$$\mu\ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r}, \tag{11}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{l^2}{2\mu r^2}. \tag{12}$$

Thus, we have reduced the problem to 1 variable. Do you notice the similarity with the approach we took in Section 2.6 (lectures 7-8) where we dealt with the bead on rotating hoop and the spherical pendulum?

As before the alternative would be to work with conservation of energy. This follows from the third symmetry, i.e no explicit time dependence of L - time translation invariance. This implies that total energy,

$$E = \frac{\mu\dot{r}^2}{2} + V_{\text{eff}}(r), \tag{13}$$

is conserved.

So far we have not made any explicit assumption about the force, apart from the fact that it is central. For gravitational force (and attractive Coulomb force) we write $V(r) = -k/r$, where $k > 0$. Hence we obtain an explicit form of the effective potential

$$V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} - \frac{k}{r}. \tag{14}$$

We can now analyse different types of orbits, e.g. closed (circular/elliptic) and open (parabolic and hyperbolic) and derive Kepler's Laws.

Let us summarize the main spirit of the above analysis. We started with 6 degrees of freedom $(\boldsymbol{r}_1, \boldsymbol{r}_2)$. Considering motion relative to the CoM reduced this to 3 degrees of freedom, $(\boldsymbol{r})$. Isotropy of space reduced this to 2 degrees of freedom, (r, ϕ), and then to one, (r). Then, the problem with 1 degree of freedom can be solved using time translational invariance (conservation of E).

---

[1] see Appendix in the end of the handout.

1 Appendix

In fact, we can obtain the same expression for the total angular momentum of the system about the centre of mass, $\boldsymbol{L}_{CoM} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$, from basic definitions. It is simple to show that $\boldsymbol{L}$, the total angular momentum of the system about the chosen fixed origin, and $\boldsymbol{L}_{CoM}$, are related by

$$\boldsymbol{L}_{CoM} = \boldsymbol{L} - \boldsymbol{R} \times \boldsymbol{P}, \tag{15}$$

where $\boldsymbol{P}$ is the total linear momentum of the system. You might remember proving this in Example Sheet 3, 1A Dynamics& Relativity. To do this write $\boldsymbol{L}_{CoM} = m_1(\boldsymbol{r}_1 - \boldsymbol{R}) \times (\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{R}}) + m_2(\boldsymbol{r}_2 - \boldsymbol{R}) \times (\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{R}})$ and expand cancelling two terms.
Now let us write (15) explicitly and simplify:

$$\boldsymbol{L}_{CoM} = m_1\boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + m_2\boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2 - \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \times (m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2) = ... = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}}. \tag{16}$$