4 Second-order ODE in the complex plane

(We start with what is essentially a revision of IA DE using the language of complex variables.)

Consider

$$w'' + p(z)w' + q(z)w = 0, \tag{1}$$

where $p(z)$, $q(z)$ and $w(z)$ are meromorphic on $\mathbb{C}$.

4.1 Classification of singular points

(a) The point $z = z_0$ is an *ordinary point* (OP) of (1) if p and q are both analytic at z_0. Otherwise, z_0 is a *singular point* (SP).

(b) If z_0 is a SP, but $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are analytic at z_0, then z_0 is a *regular singular point* (RSP). Otherwise, z_0 is an *irregular singular point.*

There exist two linearly independent solutions around RSPs.

For linear ODEs the singularities of the solutions are independent of the ICs – they are fully determined by p and q.
This does not hold for non-linear ODEs. For example,

$$w'' + w^2 = 0 \quad \Longrightarrow \quad \frac{dw}{w^2} = -dz \quad \Longrightarrow \quad w(z) = (z - z_0)^{-1},$$

where the singularity at z_0 is movable – it depends on constants of integration.

<u>What about $z = \infty$?</u> We can extend the definitions (a) and (b) by setting $z = 1/t$ and considering $t = 0$ as follows.