Classical Dynamics: Example Sheet 4

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Comments welcome: please send them to Berry Groisman (bg268@)

1. Verify the Jacobi identity for Poisson brackets:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0\,. \qquad (1)$$

2. A particle with mass m, position $\mathbf{x}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{x}\times\mathbf{p}$. Evaluate $\{x_j\,,L_k\}$, $\{p_j\,,L_k\}$, $\{L_j\,,L_k\}$ and $\{L_i\,,\mathbf{L}^2\}$.

The Runge-Lenz vector is defined as

$$\mathbf{A} \;=\; \frac{\mathbf{p}\times\mathbf{L}}{m} \;-\; \hat{\mathbf{r}}\ ,$$

where $\hat{\mathbf{r}} \;=\; \mathbf{r}/|\mathbf{r}|$. Show that $\{L_a\,,\,A_b\} \;=\; \epsilon_{abc}\,A_c$. For a system described by the Hamiltonian $H=(p^2/2m)-(1/r)$, show, using Poisson brackets, that $\mathbf{A}$ is conserved.

3. A particle of charge e moves in a background magnetic field, $\mathbf{B}$. Show that

$$\{m\,\dot{r}_a\,,\,m\,\dot{r}_b\} \;=\; e\,\epsilon_{abc}\,B_c \qquad , \qquad \{m\,\dot{r}_a\,,\,r_b\} \;=\; -\,\delta_{ab}\,.$$

A magnetic monopole is a particle which produces a radial magnetic field, of the form

$$\mathbf{B} \;=\; g\,\frac{\hat{\mathbf{r}}}{r^2}\ ,$$

where $\hat{\mathbf{r}}$ is the unit vector in the $\mathbf{r}$–direction. Consider a charged particle, moving in the background of the magnetic monopole. Define the generalised angular momentum, $\mathbf{J}=m\,\mathbf{r}\times\dot{\mathbf{r}}-eg\,\hat{\mathbf{r}}$. Show that $\{H\,,\,\mathbf{J}\}=0$. Why does this imply that $\mathbf{J}$ is conserved?

4. In the lectures we constructed canonical transformations using generating functions. Consider canonical transformations, $q_i \;\rightarrow\; Q_i(q,p)$, $p_i \;\rightarrow\; P_i(q,p)$, from the following perspective.Define $2n$-dimensional vector $\mathbf{x}=(q_1,...,q_n,p_1,...,p_n)^T$ and the $2n\times 2n$ matrix

$$J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (2)$$

where each entry is itself an $n\times n$ matrix.

i) Write Hamilton's equations for $\dot{\mathbf{x}}$ in terms of J and the Hamiltonian, H.

ii) Hence deduce the following equation for vector $\mathbf{y} = (Q_1, ..., Q_n, P_1, ..., P_n)^T$

$$\dot{\mathbf{y}} = \left(\mathcal{J} J \mathcal{J}^T\right) \frac{\partial H}{\partial \mathbf{y}}, \tag{3}$$

where $\mathcal{J}_{ij} = \partial y_i / \partial x_j$ $(i = 1, ..., 2n)$ is the Jacobian. This implies that if Jacobian of a transformation satisfies

$$\mathcal{J} J \mathcal{J}^T = J. \tag{4}$$

then Hamilton's equations are invariant under that transformation. The transformations with such a Jacobian (said to be *symplectic*) are canonical.

iii) Use the above conclusion to prove that if the Poisson bracket structure is preserved, then the transformation is canonical.

5. Show that the following transformations are canonical:

(a) $P = \frac{1}{2}(p^2 + q^2)$ and $Q = \tan^{-1}(q/p)$.

(b) $P = q^{-1}$ and $Q = p\, q^2$.

(c) $P = 2\sqrt{q}\, \left(1 + \sqrt{q}\, \cos p\right)\, \sin p$ and $Q = \log\left(1 + \sqrt{q}\, \cos p\right)$.

6. Show that the following transformation is canonical, for any constant, λ :

$$\begin{aligned} q_1 &= Q_1 \cos\lambda + P_2 \sin\lambda \quad , \quad & q_2 &= Q_2 \cos\lambda + P_1 \sin\lambda \,, \\ p_1 &= -\, Q_2 \sin\lambda + P_1 \cos\lambda \quad , \quad & p_2 &= -\, Q_1 \sin\lambda + P_2 \cos\lambda \,. \end{aligned} \tag{5}$$

Given that the original Hamiltonian is $H(q_i\,, p_i) = \frac{1}{2}({q_1}^2 + {q_2}^2 + {p_1}^2 + {p_2}^2)$, determine the new Hamiltonian, $H(Q_i\,, P_i)$. Hence, solve for the dynamics, subject to the constraint $Q_2 = P_2 = 0$.

7. A group of particles, all of the same mass m, have initial heights z and vertical momenta p, lying in the rectangle $-a \le z \le a$ and $-b \le p \le b$. The particles fall freely in the Earth's gravitational field for a time t. Find the region in phase space in which they lie at time t, and show by direct calculation that its area is still $4ab$.

8. A large, fixed number of non-interacting particles, of mass m, move in one dimension in a potential, $V(x) = \frac{1}{2} m\, \omega^2 x^2$. At time t, the number density of particles in $(x\,,\,p)$ phase space is $f(x\,,p\,,\,t)$. Initially, ω takes the value ω_1 , and particles are injected so that the number density is a constant, $f\;=\;f_1$, for all particles whose energy is less than E_1 . No particles of energy greater than E_1 are injected. How many particles are present?

The frequency of oscillation is now changed to a different value, ω_2 , so slowly that a particle's final energy does not depend appreciably on the phase of that particle in its oscillation. Use the existence of an adiabatic invariant to show that the area of phase space occupied by the particles remains unchanged. [Comment: obviously, it is quite easy to prove Liouville's theorem without involving adiabatic invariants. The point of this exercise is to illustrate the connection between the two.]

9. Explain what is meant by an *adiabatic invariant* for a mechanical system with one degree of freedom.

A light string passes through a small hole in the roof of a lift compartment of a very high skyscraper, and a small weight is attached at the lower end. Initially, the lift is at rest and the system behaves like a simple pendulum executing small oscillations. Construct a Hamiltonian for the system and use the theory of adiabatic invariants to discuss what happens to the frequency, linear and angular amplitudes of the motion if:

(a) the lift begins to move upwards slowly with constant speed, with the string attached at the hole,

(b) the lift stays at rest, but the string is slowly withdrawn through the roof.

10. Consider a system with Hamiltonian

$$H(p\,,\,q) \;=\; \frac{p^2}{2m} \,+\, \lambda\, q^{2n}\,, \tag{6}$$

where λ is a positive constant and n is a positive integer. Show that the action variable, I , and the energy, E , are related by

$$E \;=\; \lambda^{1/(n+1)} \left(\frac{n\pi I}{J_n}\right)^{2n/(n+1)} \left(\frac{1}{2m}\right)^{n/(n+1)}\,, \tag{7}$$

where $J_n \;=\; \int_0^1 (1-x)^{1/2}\, x^{(1-2n)/2n}\, dx$.

Consider a particle which moves in a potential $V(q) = \lambda\, q^4$. Assuming that λ varies slowly with time, show that the particle's total energy, E, is proportional to $\lambda^{1/3}$. Conversely, in the case that λ is fixed, show that the period of the motion is proportional to $(\lambda\, E)^{-1/4}$.

11. A pulsar, of mass m, moves in a plane orbit around a luminous supergiant star with mass $M \gg m$. You may regard the supergiant as being fixed at the origin of a plane polar co–ordinate system, $(r\,,\,\theta)$, and the neutron star as moving under a central potential, $V(r) \;=\; -\,GMm/r$. Construct the Hamiltonian for the motion, and show that p_θ and E are constants, where E is the total energy.

The neutron star is in a non–circular orbit, with $E < 0$. Give an expression for the adiabatic invariant, $J(E\,,\,p_\theta\,,\,M)$, associated with the radial motion. The supergiant is steadily losing mass in a radiatively–driven wind. Show that, over a long time–scale, one has $E \,\propto\, M^2$.

Eventually, the supergiant becomes a supernova, throwing off its outer layers on a short time–scale, and leaving behind a remnant black hole, of mass $\frac{1}{2}\,M$. Explain why the theory of adiabatic invariants cannot be used to calculate the new orbit.

Note: You may find the following integral helpful:

$$\int_{r_1}^{r_2} \left\{ \left(1 - \frac{r_1}{r}\right)\left(\frac{r_2}{r} - 1\right) \right\}^{\frac{1}{2}} dr \;=\; \frac{\pi}{2}\,(r_1 + r_2) - \pi\,\sqrt{r_1\, r_2}\;, \tag{8}$$

where $0 < r_1 < r_2$.

12.* (after 2010/Paper 4/section II/15D)

A system is described by the Hamiltonian, $H(q\,,\,p\,,\,t)$. Define the *Poisson bracket*, $\{f\,,\,g\}$, of two functions $f(q\,,\,p\,,\,t)$, $g(q\,,\,p\,,\,t)$. Show from Hamilton's equations that

$$\frac{df}{dt} \;=\; \{f\,,\,H\} \;+\; \frac{\partial f}{\partial t} \;.$$

Consider the Hamiltonian

$$H \;=\; \frac{1}{2}\,(p^2 + \omega^2\,q^2),$$

where $\omega \;=\; \omega(t)$, and define

$$a \;=\; (p \,-\, i\,\omega q)/(2\omega)^{\frac{1}{2}} \quad , \quad a^* \;=\; (p \,+\, i\,\omega q)/(2\omega)^{\frac{1}{2}} \;,$$

where $i^2 \;=\; -1$. Evaluate $\{a\,,\,a\}$ and $\{a\,,\,a^*\}$, and show that $\{a\,,\,H\} \;=\; -\,i\,\omega a$ and that $\{a^*\,,\,H\} \;=\; i\,\omega a^*$. Show further that, when $f(q\,,\,p\,,\,t)$ is regarded as a function of the independent complex variables, $(a\,,\,a^*)$, and of t, one has

$$\frac{df}{dt} \;=\; i\,\omega\left(a^*\,\frac{\partial f}{\partial a^*} \;-\; a\,\frac{\partial f}{\partial a}\right) - \;\frac{1}{2}\;\frac{\dot{\omega}}{\omega}\;\left(a\,\frac{\partial f}{\partial a^*} \;+\; a^*\,\frac{\partial f}{\partial a}\right) + \;\frac{\partial f}{\partial t} \;.$$

Deduce that, in the case $d\omega/dt \;=\; 0$, both $(\log a^* \;-\; i\,\omega t)$ and $(\log a \;+\; i\,\omega t)$ are constant during the motion.

Consider now the case in which $\omega(t)$ varies slowly with time. Writing $f \;=\; (H/\omega)$, show that the time average of (df/dt) over one period, $(2\pi/\omega)$, is approximately zero (that is, to order $(\dot{\omega}^2, \ddot{\omega})$).