General elliptic ODE (example)

Consider $(w')^2 = (w-1)(w-b)(w-c)(w-d)$. This is solved by an elliptic integral

$$z + \alpha = \int_w^\infty \frac{ds}{[(s-a)(s-b)(s-c)(s-d)]^{1/2}}. \tag{4}$$

We can reduce it to the form of eq. (3) by moving one root to ∞ and shifting the remaining roots as follows.

Consider

$$\frac{du}{[(u-a)(u-b)(u-c)(u-d)]^{1/2}}.$$

Shifting $\tilde{u} = u - d$ yields

$$\frac{d\tilde{u}}{[\,(\tilde{u}-\tilde{a})(\tilde{u}-\tilde{b})(\tilde{u}-\tilde{c})\tilde{u}\,]^{1/2}}.$$

Then $t = 1/\tilde{u}$ gives

$$\frac{-dt}{t^2[\,(\frac{1}{t}-\tilde{a})(\frac{1}{t}-\tilde{b})(\frac{1}{t}-\tilde{c})\frac{1}{t}\,]^{1/2}} = \frac{A\,dt}{[\,(t-t_1)(t-t_2)(t-t_3)\,]^{1/2}},$$

where A is a constant.

Thus, the *quartic* integrand has been reduces to a *cubic* with one root moved to ∞ (the original d is now $t = \infty$). Finally, set $t = s + \gamma$, with γ chosen such that $t_1 + t_2 + t_3$ will move to 0, then rescale s such that the leading term has coefficient 4. This will reduce Eq. (4) to Eq. (3).

Euler's Top: Consider

$$\begin{aligned} w_1' &= w_2 w_3, \\ w_2' &= w_3 w_1, \\ w_3' &= w_1 w_2, \end{aligned}$$

which gives

$$\begin{aligned} w_2' w_2 - w_3' w_3 &= 0, \quad \text{so} \quad w_3^2 = w_2^2 + B, \\ w_1' w_1 - w_3' w_3 &= 0, \quad \text{so} \quad w_3^2 = w_1^2 + C, \end{aligned}$$

(for constant B and C) and

$$w_3'^2 = (w_3^2 - B)(w_3^2 - C),$$

which is a special case of the elliptic ODE.