Classical Dynamics **Michaelmas 2014**

The derivation below is an elaborated version of the corresponding chapter from Landau-Lifshitz. This handout is available at http://www.damtp.cam.ac.uk/user/bg268/
Questions and corrections are welcome: write to Berry Groisman on bg268@

Euler Top (free asymmetric top): solution of Euler's equations in terms of elliptic integrals

Let us start with rewriting the Euler's equations

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \tag{1}$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 \tag{2}$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \tag{3}$$

and the two conserved quantities: the rotational kinetic energy and the angular momentum

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T \tag{4}$$
$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \boldsymbol{L}^2 \tag{5}$$

During the lecture we used Eqns. (4),(5) in order to introduce a nice geometic qualitative representation of the solutions - the Poinsot Construction.

Now, we are going to derive the solution of Euler's equations in terms of the elliptic integrals. (Let us assume, without loss of generality, that $I_1 < I_2 < I_3$.)
First, from (5) obtain

$$I_3\omega_3^2 = \frac{\boldsymbol{L}^2}{I_3} - \frac{I_1^2}{I_3}\omega_1^2 - \frac{I_2^2}{I_3}\omega_2^2 \tag{6}$$

and substitute it into Eq. (4) to eliminate ω_3. We can now express ω_1 in terms of ω_2, I_a, L^2 and T:

$$\omega_1^2 = \frac{1}{I_1(I_3 - I_1)}[2TI_3 - L^2 - I_2(I_3 - I_2)\omega_2^2]. \tag{7}$$

Similarly, by eliminating ω_1 we obtain

$$\omega_3^2 = \frac{1}{I_3(I_3 - I_1)}[L^2 - 2TI_1 - I_2(I_2 - I_1)\omega_2^2]. \tag{8}$$

Now substitute the expressions for ω_1 and ω_3 into Eqn. (2):

$$\dot{\omega}_2 = \frac{1}{I_2\sqrt{I_1 I_3}}\left([2TI_3 - L^2 - I_2(I_3 - I_2)\omega_2^2] \times [L^2 - 2TI_1 - I_2(I_2 - I_1)\omega_2^2]\right)^{1/2}. \tag{9}$$

As it will become evident below, the choice of the component to work with, i.e. ω_2, is related to the fact that I_2 was chosen to be the principal moment of the intermediate value.

Let us factor out the first two terms in both brackets, i.e. $2TI_3 - L^2$ and $L^2 - 2TI_1$ respectively:

$$\dot{\omega}_2 = \frac{\sqrt{(2TI_3 - L^2)(L^2 - 2TI_1)}}{I_2\sqrt{I_1 I_3}}\left[1 - \frac{I_2(I_3 - I_2)}{2TI_3 - L^2}\omega_2^2\right]^{1/2}\left[1 - \frac{I_2(I_2 - I_1)}{L^2 - 2TI_1}\omega_2^2\right]^{1/2}. \tag{10}$$

We introduce three new variables starting with

$$s^2 = \frac{I_2(I_3 - I_2)}{2TI_3 - L^2}\omega_2^2, \tag{11}$$

positivity of which follows from our choice of the balance of the principal moments of inertia. Remember from Poisont construction, that for the solution to exist the magnitude of the angular momentum has to be bounded by the semi-axes of the intertia ellipsoid, i.e. $2TI_1 < L^2 < 2TI_3$? Note, that if we were to choose $I_3 < I_2 < I_1$ the RHS in (11) still would be positive.

Thus,

$$\dot{\omega}_2 = \frac{\sqrt{(2TI_3 - L^2)(L^2 - 2TI_1)}}{I_2\sqrt{I_1 I_3}}[1 - s^2]^{1/2}\left[1 - \frac{I_2(I_2 - I_1)}{L^2 - 2TI_1}\omega_2^2\right]^{1/2}. \tag{12}$$

Now we substitute for $\dot{\omega}_2$ and ω_2^2 in terms of s and introduce the second variable

$$k^2 = \frac{(I_2 - I_1)(2TI_3 - L^2)}{(I_3 - I_2)(L^2 - 2TI_1)} > 0, \tag{13}$$

where we use the fact that $I_1 < I_2 < I_3$ and $2TI_1 < L^2 < 2TI_3$ again.

We obtain

$$\sqrt{\frac{2TI_3 - L^2}{I_2(I_3 - I_2)}}\,\frac{ds}{dt} = \frac{\sqrt{(2TI_3 - L^2)(L^2 - 2TI_1)}}{I_2\sqrt{I_1 I_3}}\sqrt{[1 - s^2][1 - k^2 s^2]}, \tag{14}$$

which is simply

$$\frac{ds}{dt} = \frac{\sqrt{(L^2 - 2TI_1)(I_3 - I_2)}}{\sqrt{I_1 I_2 I_3}}\sqrt{[1 - s^2][1 - k^2 s^2]}. \tag{15}$$

What is left is to introduce a new variable for time,

$$\tau = \sqrt{\frac{(L^2 - 2TI_1)(I_3 - I_2)}{\sqrt{I_1 I_2 I_3}}}\,dt \tag{16}$$

and finally obtain the *Jacobi elliptic integral*

$$\tau = \int_0^s \frac{ds}{\sqrt{[1 - s^2][1 - k^2 s^2]}}, \tag{17}$$

where we have chosen $t = 0$ when $\omega_2 = 0$.

There is one issue we need to resolve, namely that we should assertain that in the elliptic integral $k^2 < 1$. Let us return to Eqn. (13). If $L^2 > 2TI_2$ then it is not difficult to see that the above condition is satisfied. This is because

$$(I_2 - I_1)(2TI_3 - L^2) < 2T(I_2 - I_1)(I_3 - I_2), \tag{18}$$
$$(I_3 - I_2)(L^2 - 2TI_1) > 2T(I_3 - I_2)(I_2 - I_1). \tag{19}$$

If $L^2 < 2TI_2$, then we could *rename* the moments I_1 and I_3, in which case we simply need to interchange them in all the above formulae. It was already mentioned that such interchange will not change the sign of s^2 and k^2. Thus, let us assume, without loss of generality that $L^2 > 2TI_2$.

Inverting (17) gives us s as a Jacobian elliptic function of τ:

$$s = sn(\tau, k) \tag{20}$$

Thus, from Eqn. (11)

$$\omega_2 = \sqrt{\frac{2TI_3 - L^2}{I_2(I_3 - I_2)}} sn(\tau, k), \tag{21}$$

Using the definitions of the other two elliptic functions, namely $cn(\tau, k) = \sqrt{1 - sn^2(\tau, k)}$ and $dn(\tau, k) = \sqrt{1 - k^2 sn^2(\tau, k)}$ we obtain the relations for the two remaining components of $\boldsymbol{\omega}$.

$$\omega_1 = \sqrt{\frac{2TI_3 - L^2}{I_1(I_3 - I_1)}} sn(\tau, k), \tag{22}$$

$$\omega_3 = \sqrt{\frac{L^2 - 2TI_1}{I_3(I_3 - I_1)}} dn(\tau, k). \tag{23}$$

Functions (21), (22) and (23) are periodic. The period in τ equals $4K$, where K is the complete elliptic integral

$$K = \int_0^1 \frac{ds}{\sqrt{[1 - s^2][1 - k^2 s^2]}}. \tag{24}$$

Thus, the period in t is

$$T = 4K \sqrt{\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2TI_1)}}. \tag{25}$$

After this time the vector $\boldsymbol{\omega}$ returns to its initial position relative to the body frame. The top itself is not in the same position relative to the space frame, though!

Let us quickly test the obtained solution for the components of $\boldsymbol{\omega}$ by applying it first to the symmetric top with $I_1 = I_2$. Clearly,

$$\begin{aligned} & k^2 = 0, \\ & sn(\tau, 0) = \sin \tau, \\ & cn(\tau, 0) = \cos \tau, \\ & dn(\tau, 0) = 1, \end{aligned} \tag{26}$$

which is consistent with the results we have obtained for the symmetric top.

Second, if $L^2 = 2TI_3$, then $\omega_1 = \omega_2 = 0$, while ω_3 is constant. This corresponds to uniform rotation about axis $\mathbf{e}_3$ as we have seen before. The similar behaviour is observed when $L^2 = 2TI_1$.