Hypergeometric function - series representation

Claim:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(x)_n = x(x+1)...(x+n-1)$, $(x)_0$=1, is the *Pochhammer symbol.*

The series is convergent in $|z| < 1$. The RSP at $z = 1$ prohibits the series from converging further.

Proof: Since the solution $w(z)$ is analytic at $z = 0$, it can be written as a Taylor series

$$w(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Substitute it in (†) (the HGE labelled in the lecture) and multiply by $z - 1$ to get

$$\begin{aligned}[n(n-1)a_n z^{n-2} &+ (n+1)na_{n+1}z^{n-1} + cna_n z^{n-2} + c(n+1)a_{n+1}z^{n-1} \\ &+ (1+a+b-c)(na_n z^{n-1} + (n+1)a_{n+1}z^n) + ab(a_n z^{n-1} + a_{n+1}z^n)] = 0.\end{aligned}$$

The coefficient of z^{n-1} gives

$$\begin{aligned}[n(n-1)a_n - (n+1)na_{n+1} + cna_n - c(n+1)a_{n+1} \\ + (1+a+b-c)na_n + aba_n = 0.\end{aligned}$$

and thus

$$a_n(n^2 + an + bn + ab) = a_{n+1}(n+1)(c+n).$$

Hence,

$$a_{n+1} = \frac{(a+n)(b+n)}{(c+n)} \frac{1}{n+1} a_n,$$

which gives the required result.

Many special functions are special cases of HGF. For example,

$$\begin{aligned}(1-z)^n &= F(-n,1;1;z), \\ \log(1-z) &= zF(1,1;2;z), \\ \exp(z) &= \lim_{b\to 0} F(1,b;1;z/b).\end{aligned}$$

Integral representation of the HGF

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,$$

for $Re\, c > Re\, b > 0$ and $|z| < 1$. (See Example Sheet 4.)

The above conditions are needed for the integral to converge at its branch points $0, 1, 1/z$.