Lecture 15 (11th of November 2014)

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3.5 Euler Angles

It is time to introduce explicit parametrization of the rotation matrix R, which rotates the space frame $\{\tilde{\boldsymbol{e}}_a\}$ to the body frame $\{\boldsymbol{e}_a\}$, i.e. $\boldsymbol{e}_a = R_{ba}\tilde{\boldsymbol{e}}_b$. As it was mentioned earlier, R is a real orthogonal matrix and hence requires three parameters. We need to impose a stronger condition on R, however, namely that it should correspond to a physical rotation ($\det R = 1$) and not reflection ($\det R = -1$). This is achieved by the following explicit construction. The overall rotation is decomposed into three consecutive rotations of the space frame. We are starting with the body frame, which is aligned (coincides) with the space frame and perform three consecutive rotations. (Below we use the following notation: $R(\hat{\mathbf{a}}, \alpha)$ is a rotation by angle α about unit axis $\hat{\mathbf{a}}$.)

(1) $R(\tilde{\mathbf{e}}_3, \phi)$, i.e. rotation by angle ϕ about the third axis of the space frame, $\tilde{\boldsymbol{e}}_3$. This results in the new frame $\{\boldsymbol{e}'_a\}$. (Notice that $\mathbf{e}'_3 = \tilde{\mathbf{e}}_3$.)

(2) $R(\boldsymbol{e}'_1, \theta)$, i.e. rotation by angle θ about the axis $\boldsymbol{e}'_1$ of the new frame. We obtain a new frame, $\{\boldsymbol{e}''_a\}$ as a result. (Notice that $\mathbf{e}''_1 = \mathbf{e}'_1$.)

(3) $R(\boldsymbol{e}''_3, \psi)$, i.e. rotation by angle ψ about the axis $\boldsymbol{e}''_3$ of the new frame. (In fact $\boldsymbol{e}''_3$ is already the axis $\boldsymbol{e}_3$ of the body frame.) This accomplishes the entire rotation.

The three rotations are characterised by the following three rotation matrices.

$$R(\boldsymbol{e}''_3,\psi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; R(\boldsymbol{e}'_1,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \; R(\tilde{\mathbf{e}}_3,\phi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

The resulting matrix is

$$R = R(\boldsymbol{e}''_3,\psi)R(\boldsymbol{e}'_1,\theta)R(\tilde{\boldsymbol{e}}_3,\phi). \tag{2}$$

It might not be immediately obvious how to calculate the above product, because each matrix describes rotation in a new frame. It is possible to show, that (see handout - non-examinable), that it can be replaced by rotation matrices about the original space-axes $\{\tilde{\boldsymbol{e}}_a\}$ multiplied in the reversted order, i.e.

$$R = R(\tilde{\mathbf{e}}_3,\phi)R(\tilde{\mathbf{e}}_1,\theta)R(\tilde{\mathbf{e}}_3,\psi) \tag{3}$$

Now, the matrices can be multiplied in a normal way, which yields

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\sin\psi\cos\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}. \tag{4}$$

The transposed matrix

$$R^\intercal = (R(\tilde{\mathbf{e}}_3,\phi)R(\tilde{\mathbf{e}}_1,\theta)R(\tilde{\mathbf{e}}_3,\psi))^\intercal = R^\intercal(\tilde{\mathbf{e}}_3,\psi)R^\intercal(\tilde{\mathbf{e}}_1,\theta)R^\intercal(\tilde{\mathbf{e}}_3,\phi)$$

describes how components of a vector are changed due to the change of frame.

The angles ϕ, θ and ψ play the role of generalised coordinates in the parametrization of configuration of a rigid body and known as *Euler Angles*.

We can write $\boldsymbol{\omega}$ in body frame and in space frame using Euler Angles. To this end we recall that $\boldsymbol{\omega}$ is instantaneous angular velocity and therefore

$$\boldsymbol{\omega} = \dot{\phi}\tilde{\boldsymbol{e}}_3 + \dot{\theta}\boldsymbol{e}_1' + \dot{\psi}\boldsymbol{e}_3. \tag{5}$$

If we want to write $\boldsymbol{\omega}$ in the body frame, then we expand $\tilde{\boldsymbol{e}}_3$, $\boldsymbol{e}_1'$ in the body frame basis and substitute. Similarly, if we want to write $\boldsymbol{\omega}$ in the body frame, then we expand $\boldsymbol{e}_1'$, $\boldsymbol{e}_3$, in the space frame basis and substitute.
Let us write components of $\boldsymbol{\omega}$ in the body frame. We could work with the matrix R. This is what you are expected to do in question 2, example sheet 3 (there you will also write the angular velocity in the space frame). Here we will simply inspect the diagram sketched in the class, which shows that

$$\tilde{\boldsymbol{e}}_3 = \sin\theta(\sin\psi\boldsymbol{e}_1 + \cos\psi\boldsymbol{e}_2) + \cos\theta\boldsymbol{e}_3, \qquad \boldsymbol{e}_1' = \cos\psi\boldsymbol{e}_1 - \sin\psi\boldsymbol{e}_2. \tag{6}$$

As a result

$$\boldsymbol{\omega} = [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\mathbf{e}_1 + [\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]\mathbf{e}_2 + [\dot{\psi} + \dot{\phi}\cos\theta]\mathbf{e}_3 \tag{7}$$

First application of Euler Angles: calculating angular frequency of precession of $\mathbf{e}_3$ about $\tilde{\mathbf{e}}_3$ for free symmetric top ($\dot{\phi}$). Recall that $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$ is conserved. Also, $L_3 = I_3\omega_3$ is conserved. θ is the angle between $\boldsymbol{L}$ and $\mathbf{e}_3$, so $\dot{\theta} = 0$.

In addition, we note that

$$\dot{\psi} = -\Omega = \omega_3\frac{I_1 - I_3}{I_1}. \tag{8}$$

Hence,

$$\dot{\phi} = \frac{\omega_3}{\cos\theta}\frac{I_3}{I_1}, \tag{9}$$

and we conclude that $\dot{\phi}$ is constant.

3.6 Heavy Symmetric Top (known as Lagrange Top)

Armed with the parametrization of $\boldsymbol{\omega}$ in terms of Euler angles, we can analyse the problem of rotation of a symmetric top pinned at a point, P (which lies on the axis of symmetry of the top), and acted upon by gravity. We assume that $I_1 = I_2 \neq I_3$ are known [1]. Also we are given its mass, M, and the distance, l, of the centre of mass from P.
We write the Lagrangian of the top (assuming that $\{\mathbf{e}_a\}$ are chosen to be the principal axes),

$$L = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 - Mgl\cos\theta, \tag{1}$$

and use Euler angles to obtain

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{I_3}{2}(\dot{\psi} + \cos\theta\dot{\phi})^2 - Mgl\cos\theta. \tag{2}$$

There is no explicit time dependence in L and ϕ, ψ are ignorable, so there are three conserved quantities.

(a) $p_\psi = \frac{\partial L}{\partial\dot{\psi}} = I_3(\dot{\psi} + \cos\theta\dot{\phi}) = I_3\omega_3$. Hence, as with free symmetric top, ω_3 - the *spin* - is conserved.

(b) $p_\phi = \frac{\partial L}{\partial\dot{\phi}} = I_1\sin^2\theta\dot{\phi} + I_3\omega_3\cos\theta$.

(c) $E = T + V = \frac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2}I_3\omega_3^2 + Mgl\cos\theta$.

[1] Strictly speaking I_1 and I_2 are meant to be given w.r.t. P. We will immediately see that due to conservation of ω_3 we can use values calculated w.r.t the centre of mass as the additional term in the Lagrangian (from parallel axis theorem) will be constant.

We will treat ω_3, p_ϕ, p_ψ, and E as fixed parameters.

Our general aim is to obtain solutions $\theta(t)$, $\phi(t)$, $\psi(t)$. Following tradition established in the literature let us define two parameters

$$a = \frac{I_3}{I_1}\omega_3, \qquad b = \frac{p_\phi}{I_1}. \tag{3}$$

Thus, (a) and (b) can be rewritten as

$$\begin{aligned} \dot{\phi}(\theta) &= \frac{b - a\cos\theta}{\sin^2\theta}, \\ \dot{\psi}(\theta) &= \frac{I_1}{I_3}a - \frac{(b - a\cos\theta)\cos\theta}{\sin^2\theta}. \end{aligned} \tag{4}$$

Therefore, if we solve for $\theta(t)$, then we can solve the above equations to obtain $\phi(t)$ and $\psi(t)$. Notice, that Lagrange Top is a generalisation of a spherical pendulum. As with spherical pendulum, we will try to reduce the problem to 1 DoF, θ. One possibility is to write E-L equation for θ and obtain

$$I_1\ddot{\theta} = -\frac{\partial V_{\text{eff}}}{\partial\theta}, \tag{5}$$

where

$$V_{\text{eff}}(\theta) = \frac{I_1(b - a\cos\theta)^2}{2\sin^2\theta} + Mgl\cos\theta. \tag{6}$$

Hopefully, this approach looks familiar! We reduced the problem to a single variable.

Alternatively, we could start with the expression for total energy, (c).

$$\tilde{E} = E - \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + Mgl\cos\theta, \tag{7}$$

where the *reduced energy* $\tilde{E}$ - total energy less rotational kinetic energy about $\mathbf{e}_3$ - is conserved as well. Substitute for $\dot{\phi}$ from (4) and obtain

$$\tilde{E} = \frac{1}{2}I_1\dot{\theta}^2 + V_{\text{eff}}(\theta). \tag{8}$$

We can now analyse the shape of $V_{\text{eff}}(\theta)$, as we did for spherical pendulum. For fixed $\tilde{E}$ the motion is confined to the range $\theta_1 \leq \theta \leq \theta_2$, where θ_1 and θ_2 are roots of $\tilde{E} = V_{\text{eff}}(\theta)$. The character of the motion depends on the signs of $\dot{\phi}$ at θ_1 and θ_2. There are three possibilities.

1. Case 1: $\dot{\phi}$ has a same sign.
2. Case 2: $\dot{\phi}$ changes sign.
3. Case 3: $\dot{\phi}$ becomes zero at θ_1. Note that it cannot be zero at θ_2, which follows from the expressions for energy and p_ϕ.

The motion in ϕ is called *precession of the top*. The motion in θ is called *nutation of the top*.

Uniform precession

Can precession be uniform with no nutation, i.e. $\dot{\theta} = 0$, while $\dot{\phi}$ is constant? Such a motion will correspond to the minimum of the potential. The equilibrium value θ_0 satisfies

$$\frac{\partial V_{\text{eff}}}{\partial\theta} = I_1 a\sin\theta_0\dot{\phi} - I_1\sin\theta_0\cos\theta_0\dot{\phi}^2 - Mgl\sin\theta_0 = 0, \tag{9}$$

which leads to a quadratic equation for $\dot{\phi}$

$$\dot{\phi} = \frac{I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4I_1\cos\theta_0 Mgl}}{2I_1\cos\theta_0}. \tag{10}$$

Thus, condition for uniform precession to exist is

$$\omega_3^2 \geq \frac{4I_1}{I_3^2} Mgl \cos\theta_0, \tag{11}$$

that is the top needs to spin fast enough about its axis of symmetry.