Lecture 18, 19 (13th and 15th of November 2012) - outline

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4 The Hamiltonian Formalism

4.1 Phase Space, Legendre Transform, Hamilton's Equations, the Printiple of Least Action

Key concepts: phase space, Hamiltonian, Legendre Transform.

The basic motivation behind Hamiltonian Formalism is to eliminate generalised velocities in favour of generalised momenta and place generalised coordinates and conjugate momenta on equal footing.
We introduced the concept of Phase space and discussed the Legendre Transform of a function. We then defined the Hamiltonian to be the Legendre transform of the Lagrangian with respect to generalised velocities $\dot{q}_i$,

$$H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t). \tag{1}$$

The recipe for eliminating $\dot{q}_i$ in the RHS is simple: we invert $p_i(q_j, \dot{q}_j, t) = \frac{\partial L}{\partial \dot{q}_i}$ to obtain $\dot{q}_i(q_j, p_j, t)$ and substitue it into (1).

We then derived Hamilton's Equations

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \\ \dot{q}_i &= \frac{\partial H}{\partial p_i}. \end{aligned} \tag{2}$$

These are $2n$ 1st order differential equations for q_i and p_i. (Compare with n 2nd order DE for q_i and $\dot{q}_i$ in Lagrangian formalism.) We derived (2) using two methods. First, by considering the differential of (1). Second, by varying the action integral. In the latter case we pointed out that, unlike in Lagrangian formalism where $\dot{q}$ varied authomatically with q, here we have to vary q and p independently. This supports our vision that q and p can be treated equally. However, there is a difference after all. We only have to require that $\delta q_i(t_1) = \delta q_i(t_2) = 0$, while p_i can be free at t_1 and t_2. To restore the symmetry between q and p we can simply impose $\delta p_i(t_1) = \delta p_i(t_2) = 0$. This will have an additional advantage of us being able to add full time derivatives of arbitrary functions $F(q, p)$ to the action inside the integral.

We then discussed two conservation laws, namely

(a) If $\frac{\partial H}{\partial t} = 0$ then H is a constant of motion. It follows trivially from $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$.

(b) If q is ignorable (cyclic) with respect to the Lagrangian, then it is also ignorable w.r.t the Hamiltonian. This follows by construction, indeed $p = \frac{\partial L(\dot{q})}{\partial \dot{q}_i} = p(\dot{q})$, since L does not depend on q.

We then considered two examples:

(1) A particle moving in 3-dimensional potential (trivial example)

(2) A charged particle in electro-magnetic field.
We derived the Hamiltonian in the form

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi.$$

One can now write Hamilton's equations and verify that they give Lorentz force law. We didn't do it explicitely. We considered a particular case with $\mathbf{A} = (-\mathbf{By}, \mathbf{0}, \mathbf{0})$, which corresponds to uniform magnetic field of magnitued B is z-direction. We have shown that for a particle which moved in (x,y)-plance the motion will be circular.

4.2 Liouville's Theorem

We stated and proved Liouville's theorem.