Lectures 20 and 21 (17th adn 20th of November 2012) - outline

Comments and questions should be sent to Berry Groisman, bg268@

4.2 Liouville's Theorem - continued

We derived Liouville's Equation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i} = 0.$$

This is an immediate consequence of Liouville's Theorem and Hamilton's Equations of motion.

4.3 Poisson Bracket

For two functions on phase space, $f(q,p,t)$ and $g(q,p,t)$ the Poisson bracket is defined as

$$\{f,g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{1}$$

Properties of Poisson brackets:

(1) Anti-commutativity: $\{f,g\} = -\{g,f\}$.

(2) Linearity: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$, $\forall\ \alpha,\beta \in \mathbb{R}$.

(3) Leibniz rule (follows from the chain rule): $\{fg,h\} = f\{g,h\} + \{f,h\}g$.

(4) Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

(5) $\forall\ f(q,p,t)$

$$\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}. \tag{2}$$

Conserved quantities: An important consequence of (5) is that if a function $I(q,p)$ on phase-space does not depend on time explicitely (i.e. $\frac{\partial I}{\partial t} = 0$) and $\{I,H\} = 0$, then I is a constant of motion.

It also follows from Jacobi identity that if I and J are constants of motion, then $\{I,J\}$ is also a constant of motion.

We discussed two examples: Angular Momentum and (Laplace-)Runge-Lenz vector.

4.4 Canonical Transformations

In Hamiltonian Formalism we can treat generalized coordinates and momenta (almost) equally. This allows for a wider class of transformations, comparing with Lagrangian formalism. We can "mix" coordinates and momenta, so to speak:

$$\begin{cases} Q_i = Q_i(q,p), \\ P_i = P_i(q,p), \end{cases} \tag{3}$$

where Q and P are $2n$ new independent variables. Of course, not any transformation will be good for us. We require that a transformation preserves canonical form of equations of motion, Hamilton's Equations, that is. Such transformations are called *canonical*.

We have derived the formulae for canonical transformation $(q, p) \to (Q, P)$ using *generating function* $F(q, Q, t)$

$$\begin{cases} p_i = \dfrac{\partial F}{\partial q_i}, \\ P_i = -\dfrac{\partial F}{\partial Q_i}, \\ H' = H + \dfrac{\partial F}{\partial t}. \end{cases} \tag{4}$$

If F does not depend on time explicitely then new Hamiltonian H can be obtained by substituting $q(Q, P)$ and $p(Q, P)$ into H.

An alternative way to introduce canonical transformations is via Jacobian of the transformation (see D. Tong, section 4.4, page 100).

We discussed the the following theorem.

Theorem 4.1 *The Poisson Bracket is unvariant under Canonical Transformations, i.e. if the transformation* $(q, p) \to (Q, P)$ *is canonical, then for two function on phase-space,* $f(q, p)$ *and* $g(q, p)$*, the following holds*

$$\{f, g\}_{q,p} = \{f, g\}_{Q,P}. \tag{5}$$

We proved sufficient condition in the lecture and left the necessary for the examle sheet.

We discussed a simple example of a free particle to appreciate the power of CT. In that example the role of spacial coordinate and momentum were swapped. Thus the more general class of transformation allowed in Hamiltonian Formalism effectively removes the boundary betweem q and p. From now on we will call them simply *canonical conjugate variables*.