# Lectures 22 and 23 (22th and 24th of November 2012) - outline

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## 4.5 Action and Angle Variables

We have seen that Hamiltonian Formalism allows much wider class of coordinate transformations. A suitable choice of new conjugate variables can simplify the situation drastically. In many cases there is a natural choice of variables, which simplify the problem - they are known as "action-angle variables".
We introduced the idea using the simple example of

### 4.5.1 Simple Harmonic Oscillator in 1-d

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{1}$$

For fixed total energy, $E$, the trajectories in phase-space are ellipses with semi-axes $\sqrt{2E/m\omega^2}$ and $2mE$.

If we choose new conjugate variables, $(\theta, I)$ as follows:

$$\begin{cases} q = \sqrt{\frac{2I}{m\omega}}\sin\theta, \\ p = \sqrt{2Im\omega}\cos\theta, \end{cases} \tag{2}$$

then it is easy to show that $H' = \omega I$, and from Hamilton's equations it follows, that $\dot{\theta} = \omega$, $\dot{I} = 0$. Thus, the motion in phase space becomes trivial - flow paths are straight lines with constant $I$. Variables $(\theta, I)$ are called *angle and action variables.*

We then stated (without proof) the (Arnold-)Liouville's Theorem for Integrable Systems:

**Theorem 4.1** *If $\exists$ $n$ constants of motion $O_i$, $i = 1, 2, ..., n$, s.t. $\{O_i, O_j\} = 0$, then angle-action variables exist and the system is integrable.*

### 4.5.2 Angle-Action Variables for closed orbits in 2-d phase-space

Consider a more general example of a 1-d system with potential $V(q)$, which has (local) minimum. The trajectories in phase-space will be closed orbits with constant total energy. We conjectured and proved that the correct choice for the actions variable is

$$I(E) = \frac{1}{2\pi}\oint pdq, \tag{3}$$

that is the area of phase-space enclosed by an orbit (multiplied by $1/2\pi$). It is a function of $E$ only.
The angle variable can be calculated as

$$\theta = \frac{d}{dI}\int pdq. \tag{4}$$

We note that all 1-d systems are integrable because $E$ is conserved.