Lecture 22 (27th of November 2014) - outline

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(This lecture will be slightly updated once we finish lecture 19.)

4.5 Action and Angle Variables

We have seen that Hamiltonian Formalism allows much wider class of coordinate transformations. A suitable choice of new conjugate variables can simplify the situation drastically. In many cases there is a natural choice of variables, which simplify the problem - they are known as "action-angle variables".
We introduce the idea using the example of a

4.5.1 Simple Harmonic Oscillator in 1-d

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{1}$$

For fixed total energy, E, the trajectories in phase-space are ellipses with semi-axes $\sqrt{2E/m\omega^2}$ and $\sqrt{2mE}$.

If we choose new conjugate variables, (θ, I) as follows. Consider the generating function

$$F(q,Q) = \frac{1}{2}\omega^2 q^2 \cot\theta. \tag{2}$$

Using the formulae for canonical transformations (Section 4.4) we obtain

$$\begin{aligned} p =& \frac{\partial F}{\partial q} = m\omega^2 q \cot\theta, \\ I =& -\frac{\partial F}{\partial \theta} = \frac{m\omega^2 q^2}{2\sin^2\theta}. \end{aligned} \tag{3}$$

Hence,

$$\begin{cases} q = \sqrt{\frac{2I}{m\omega}}\sin\theta, \\ p = \sqrt{2Im\omega}\cos\theta. \end{cases} \tag{4}$$

We have verified that the above transformation is canonical by explicitly showing that $\{q,q\}_{\theta,I} = \{p,p\}_{\theta,I} = 0$, $\{q,p\}_{\theta,I} = 1$.

It is easy to show that in the new variables, θ, I, the Hamiltonian is $H' = \omega I$, and from Hamilton's equations it follows, that $\dot{\theta} = \omega$, $\dot{I} = 0$. Thus, the motion in phase space becomes trivial - flow paths are straight lines with constant I. Variables (θ, I) are called *angle and action variables.*

We then state (without proof) the (Arnold-)Liouville's Theorem for Integrable Systems:

Theorem 4.1 *If $\exists$ n constants of motion h_i, $i = 1, 2, ..., n$, s.t. $\{h_i, h_j\} = 0$, then angle-action variables exist and the system is integrable.*

4.5.2 Angle-Action Variables for closed orbits in 2-d phase-space

Consider a more general example of a 1-d system with potential $V(q)$, which has (local) minimum. The trajectories in phase-space will be closed orbits with constant total energy. We conjectured and proved that the correct choice for the actions variable is

$$I(E) = \frac{1}{2\pi}\oint p dq, \tag{5}$$

that is the area of phase-space enclosed by an orbit (multiplied by $1/2\pi$). It is a function of E only.
The angle variable can be calculated as

$$\theta = \frac{d}{dI}\int p dq. \tag{6}$$

We note that all 1-d systems are integrable because E is conserved.

We discussed two ways of proving the above result. First, by direct integration of

$$dt = \sqrt{\frac{m}{2}} \frac{dq}{\sqrt{E - V(q)}} \tag{7}$$

over one orbit. Second, alternative proof uses generating function. Here I describe the latter. Consider $F(q, \theta)$. The motion is periodic in q, p, therefore it must be periodic in θ. Thus, F is periodic in θ. Recall the expression for the differential of F:

$$dF = pdq - PdQ + (H' - H)dt. \tag{8}$$

In our case $H' = H$ and we obtain

$$dF = pdq - Id\theta. \tag{9}$$

Let us integrate over a single period: q returns to its original value, while θ changes by amount, which we choose to be period 2π. From periodicity of F it follows that

$$\oint dF = 0 = \oint pdq - \oint Id\theta = \oint pdq - 2\pi I \quad \Rightarrow \quad \theta = \frac{d}{dI} \int pdq. \tag{10}$$