Lectures 23 and 24 (29th of November, 2nd of December 2014) - outline

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4.6 Adiabatic Invariants

It turns out that action variables play central role in the theory of *Adiabatic Invariants*. Assume, that Hamiltonian depends on some time-dependent parameter $\lambda(t)$, $H(q, p; \lambda(t)$. Total energy, E, is no longer conserved. However, if $\lambda(t)$ changes very slowly with time (adiabatically[1]), then there exist combinations of E and λ, which remains approximately constant. These are called adiabatic invariants (AI).

We gave a formal definition of AI [2].

Definition 4.1 *Let T be an arbitrary fixed time (say one period). $\forall\ \epsilon > 0$ consider variations $\lambda(t) \equiv \lambda(\epsilon, t)$, s.t. $\dot{\lambda} = \mathcal{O}(\epsilon)$, $\ddot{\lambda} = \mathcal{O}(\epsilon^2)$ (i.e. $\lambda(t) = h(\epsilon t)$ for some function h). We say that a quantity $I(t)$ is an adiabatic invariant of the dynamical system if for any such variations of λ we have*

$$|I(t) - I(0)| = \mathcal{O}(\epsilon), \quad \forall \quad 0 \leq t \leq \frac{T}{\epsilon}. \tag{1}$$

We postulated without proof that AI are the angle-actions variables of the system. For the purpose of finding of AI we treat λ as constant over a period and calculate the action variable as in Section 4.5. We have illustrated this idea with few examples. Two of them are given below.

4.6.1 AI of a 1-d Harmonic Oscillator

Consider the case when the frequency of the oscillator changes slowly, i.e. $\lambda(t) = \omega(t)$. We have already shown, that in such case the action variable (and hence the AI) is $I = E/\omega$. Thus, when frequency of a HO changes slowly with time the energy changes proportionally to frequency, $E \sim \omega$.

4.6.2 Example: (after Wells-Siklos's paper)

Consider the following Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\lambda}{2} q^4 = E. \tag{2}$$

First, we solve for p:

$$p = \sqrt{m(2E - \lambda q^4)}. \tag{3}$$

Then we calculate the action variable

$$I = \frac{1}{2\pi} \oint p dq = \frac{\sqrt{m}}{2\pi} \oint \sqrt{2E - \lambda q^4} dq = \frac{\sqrt{m}}{\pi} (2E)^{3/4} \lambda^{-1/4} \int_{-1}^{1} \sqrt{1 - x^4} dx. \tag{4}$$

Thus the AI is $I \sim (\frac{E^3}{\lambda})^{1/4}$, i.e. energy varies as $\lambda^{1/3}$ for slowly varying changes in λ.

4.6.3 The Two-Body Problem

This example is discussed during the lecture. Explicit calculation of the corresponding integral was not shown - see Problem 11 in Example Sheet 4, where one of the masses is much larger than the other.

4.6.4 A particle in a magnetic field

Detail discussion of this example is presented in Section 4.6.2 of David Tong's notes:
http://www.damtp.cam.ac.uk/user/tong/dynamics.html

[1] Adiabatic change corresponds to changes of energy small comparing to energy itself.
[2] See C. Wells and S. Siklos, Europ. J. of Phys., 28, 105-112 (2007).

4.7 Miscellaneous applications of Hamiltonian Formalism

4.7.1 Classical spin in a magnetic field.

Consider a particle with magnetic dipole moment $\boldsymbol{\mu}$, which encounters a region of inhomogeneous magnetic field $\boldsymbol{B}(\boldsymbol{r})$. The Hamiltonian of the particle is

$$H = \frac{\boldsymbol{p}^2}{2m} - \boldsymbol{\mu} \cdot \boldsymbol{B}, \tag{5}$$

where $-\boldsymbol{\mu} \cdot \boldsymbol{B}$ is the potential energy of $\boldsymbol{\mu}$ in $\boldsymbol{B}(\boldsymbol{r})$. Let us choose the coordinate axes, so the particle moves initially along $\hat{x}$. As particle passes through the field its momentum changes according to Hamilton's equation

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{r}} = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \boldsymbol{B}). \tag{6}$$

Let's us assume that $\boldsymbol{B}$ is nearly parallel to $\hat{z}$, with $B_z = B\ z$ for simplicity [3].
Thus, for the z-component of (6) we have

$$\dot{p}_z = \frac{\partial}{\partial z}(\mu_z \cdot B_z) = \mu_z \frac{\partial B_z}{\partial z}, \tag{7}$$

and the change in $\boldsymbol{p}$ is

$$\Delta p_z = \mu_z \frac{\partial B_z}{\partial z} T = \mu_z B T, \tag{8}$$

where T is the time the particle spends interacting with the field. Thus, the spin changes its transverse momentum, and this change is proportional to μ_z. This process is the basis for measurement of magnetic moments.

4.7.2 Motion of 2-d vortices.

Detail discussion of this example is presented in Section 4.3.3 of David Tong's notes:
http://www.damtp.cam.ac.uk/user/tong/dynamics.html

[3] It cannot be exactly parallel, otherwise $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ will not be satisfied. However, this assumption will do the job for the purposes of our discussion.