Lecture 6 (15th of October 2012) - outline

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2.3 Constraints and Generalised Coordinates (continued)

Key concepts: constraints, Lagrange multipliers, generalised velocities and momenta, conjugate quantities.

We have shown how constraints can be incorporated into Lagrangian formalism with the help of Lagrange multipliers. We introduced modified Lagrangian

$$L' = L(\mathbf{x}, \dot{\mathbf{x}},) + \lambda_\alpha f_\alpha$$

and obtained the following modification of E-L. equations, where the term in the RHS is a manifistation of the contraints:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_A} - \frac{\partial L}{\partial x_A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x_A}. \tag{8}$$

Thus, we have demonstrated that we can easily incorporate constraint forces into the Lagrangian set-up using Lagrange multipliers.

In many situations, however, we might not be interested in contraint forces, but just in the dynamics of the generalised coordinates, q_i. We proved the following theorem.

Theorem: For constrained systems we may derive the equations of motion directly in generalised coordinates, i.e. using

$$L(q_i, \dot{q}_i, t) = L\left(x_A(q_i, t), \dot{x}_A(q_i, t)\right)$$

and E-L equations written purely in terms of generalised coordinates.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

We briefly mentioned *non-holonomic constraints*.

Summary: General form of the Lagrangian Formalism

A mechanical system is described by n generalised coordinates (degrees of freedom), q_i, which define a point in a n-dimensional configuration space $\mathcal{C}$.
The time evolution is a curve in $\mathcal{C}$ governed by the Lagrangian $L(q_i, \dot{q}_i, t)$, such that q_i obey n coupled 2nd order differential equations (Euler-Lagrange equations)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

Some definitions:

- $\dot{q}_i$ - generalised velocities.
- $p_i = \frac{\partial L}{\partial \dot{q}_i}$ - generalised momenta conjugate to q_i.
- E-L eqns can be rewritten as

$$\dot{p}_i = \frac{\partial L}{\partial q_i}.$$

A note: $\dot{q}_i$ and p_i coincide with real velocities and momenta in Cartesian coordinates. This is not generally the case for other coordinates. Consider, for example, polar coordinates $\dot{\mathbf{r}} = (\dot{r}, r\dot{\theta})$. For the planar pendulum $\dot{\mathbf{r}} = (0, l\dot{\theta})$, but it is natural to choose generalised velocity as $q = \dot{\theta}$ (not $l\dot{\theta}$), which does not have a dimension of L/T. In this case generalised momentum is $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, while the real momentum is $ml\dot{\theta}$.

2.4 Noether's Theorem and Symmetries

This sections discusses appearence of conservations laws in Lagrangian formalism. In particular, we are going to state and prove Noether's Theorem, which relates conserved quantities to symmetries.

Definition: We say that $F(q_i, \dot{q}_i, t)$ is a constant of motion (conserved quantity) if

$$\frac{dF}{dt} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial F}{\partial t} = 0,$$

where q_i satisfies E-L equations.
F remains constant along the path followed by the system.

We gave two examples:

- If $\frac{\partial L}{\partial t} = 0$ then $H(q_i, p_i) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ (the Hamiltonian) is conserved.
- If $\exists\ q_j$ s.t. $\frac{\partial L}{\partial q_j} = 0$ then p_j is conserved. Such a coordinate is called *ignorable* or *cyclic*.