Lecture 7 (18th of October 2012) - outline

Comments and questions should be sent to Berry Groisman, bg268@

2.4 Noether's Theorem and Symmetries (continued)

Key concepts: Noether's Theorem.

Noether Theorem (NT) - proved by Emmy Noether in 1918:
Consider continuous one-parameter[1] family of transformation for the coordinates of the system

$$q_i(t) \to Q_i(s,t), \tag{1}$$

where $s \in R$ (continuous parameter of the transformation), such that $Q_i(0,t) = q_i(t)$.

If L is invariant under this tranformation, i.e.

$$L(Q_i(s,t), \dot{Q}_i(s,t), t) = L(q_i, \dot{q}_i, t),$$

then it must not depend on s:

$$\frac{d}{ds} L(Q_i(s,t), \dot{Q}_i(s,t), t) = 0.$$

If this is the case, then the transformation is said to be a *continuous symmetry of L.*

Theorem 2.1 *NT states that for each such symmetry there exists a conserved quantity.*

Proof (The proof constitues in deriving such a quantity and providing the recipy for its calculation.)

$$\begin{aligned} \frac{dL}{ds} &= \frac{\partial L}{\partial Q_i}\frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i}\frac{d\dot{Q}_i}{ds} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right)\frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i}\frac{d\dot{Q}_i}{ds}. \quad \text{(using E-L equations)} \end{aligned}$$

Thus,

$$\left.\frac{dL}{ds}\right|_{s=0} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\left.\frac{dQ_i}{ds}\right|_{s=0}\right) = 0.$$

Thus, there is a conserved quantity, which is

$$\sum_i p_i \left.\frac{dQ_i}{ds}\right|_{s=0}.$$

□

In other words, NT provides the recipe: the conserved quantity can be found by differentiating each coordinate with respect to the parameters of the transformation in the immediate neighbourhood of the identity transformation, multiplied by corresponding generalised momentum and summing over all degrees of freedom.

We applied NT to the symmetries corresponding to three elements of the Galilean Group (for a closed system of N interacting particles), namely

- Homogeneity of Space $\Longrightarrow$ Translational Invariance of L $\Longrightarrow$ Conservation of Total Linear Momentum (*Calculation was given in the class*)

[1] Generalisation to several parameters is trivial.

- (*The following calculation was not shown in the class*)
  Consider a rotation of a system as a whole around an arbitrary $\hat{\mathbf{n}}$. For an infinitecimal rotation the corresponding transformation of the coordinates is

$$\mathbf{r_i} \to \mathbf{r_i} + \delta\mathbf{r_i} \approx \mathbf{r_i} + s\ \hat{\mathbf{n}} \times \mathbf{r_i}, \tag{2}$$

  where s is small.

  For a closed system the Lagrangian $L = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2 - \sum_{i,j} V(|\mathbf{r}_i - \mathbf{r}_j|)$ is unvariant under such transformation

$$L(\mathbf{r_i} + s\ \hat{\mathbf{n}} \times \mathbf{r_i}, \dot{\mathbf{r}}_\mathbf{i} + s\ \hat{\mathbf{n}} \times \dot{\mathbf{r}}_\mathbf{i}, t) = L(\mathbf{r_i}, \dot{\mathbf{r}}_\mathbf{i}, t) \tag{3}$$

  The conserved quantity is calculated as

$$\sum_i \mathbf{p}_i \cdot \frac{\partial}{\partial s}(\mathbf{r_i} + s\ \hat{\mathbf{n}} \times \mathbf{r_i})\bigg|_{s=0} = \sum_i \mathbf{p}_i \cdot (\hat{\mathbf{n}} \times \mathbf{r_i}) = \hat{\mathbf{n}} \cdot \sum_i \mathbf{r_i} \times \mathbf{p}_i, \tag{4}$$

  which is total angular momentum in the direction of $\hat{\mathbf{n}}$. As $\hat{\mathbf{n}}$ is arbitrary we conclude that the total angular momentum is conserved. Thus,
  Isotropy of Space $\Longrightarrow$ Rotational Invariance of L $\Longrightarrow$ Conservation of Total Angular Momentum

- Homogeneity of Time: L is invariant under $t \to t + s$, that is $\frac{\partial L}{\partial t} = 0$, which implies (as we have shown earlier) that Hamiltonian

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L. \tag{5}$$

  is conserved.
  For a closed system (and more generally for systems with scleronomic constraints) Hamiltonian equals to the total energy, therefore we have the fundamental link

  Homogeneity of Time $\Longrightarrow$ Conservation of Total Energy (for scleronomous systems)

  It is worth noting that for some rheonomous systems Lagrangian does not depend on time explicitely, in which case H is conserved. However it is not necessarily equal to the total energy, which might not be conserved (as in the example of the bead on rotating circular hoop - see Application II in the next lecture).

2.5 Applications

Application I: Bead on a square rotating hoop *(This was described in detail during the lecture)*