## Lectures 8, 9, 10 (20th, 23rd and 25th of October 2012) - outline

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## 2.5 Applications (continued)

**Application II: Bead on rotating circular hoop** *(This was described in detail during the lecture)*
An important following remark was made. Lagrangian does not depend on time and therefore the Hamiltonian

$$H = \dot{\psi}\frac{\partial L}{\partial \dot{\psi}} - L = ma^2\left(\frac{\dot{\psi}^2}{2} - \omega^2 \sin^2\psi\right) - mga\cos\psi$$

is conserved. The last term in the RHS is obviously the potential energy. However, the first term is *not* kinetic energy. The kinetic energy is

$$T = ma^2\left(\frac{\dot{\psi}^2}{2} + \omega^2 \sin^2\psi\right).$$

This is a typical situation corresponding to *rheonomic* constraint, that is time dependent constraint, with $L$ which is not explicitely time dependent. The bead on rotating hoop is an excellent example of such a system. $H$ is conserved, but total energy is not. There is a flow of energy from outside, e.g. from the motor which rotates the hoop. In this system the change in kinetic energy of the bead does not account for the change in potential energy. If, for example, the hoop rotates very fast and the initial position of the bead is close to the bottom, then it will move up to reach the value of $\psi$ corresponding to stable equilibrium. To maintain the angular velocity of rotation constant we need to supply energy from outside. If we do not, the hoop will slow down.

**Application III: Double pendulum** (*We left it to the example sheet 1*)

**Application IV: Spherical Pendulum** *(This was described in detail during the lecture)*

**Application V: Two-body Problem** *(This was described in detail during the lecture)*

**Application VII: Restricted three-body problem** *(not discussed)*

**Application VII: Charged particle in a background electro-magnetic field** *(This was described in detail during the lecture)*

## 2.6 Quadratic Lagrangians; Small Oscillations, Stability and Normal Modes

We have cosidered a general form of the kinetic energy term in Lagrangian, which results from the transformation from Cartesian to generalised coordinates.

*Case 1 - Scleronomic constraints:* $x_A = x_A(\mathbf{q})$ and therefore

$$\dot{x}_A = \sum_i^n \frac{\partial x_A}{\partial q_i}\dot{q}_i.$$

Thus,

$$T = \frac{1}{2}\sum_A \dot{x}_A^2 = \frac{1}{2}\sum_{i,j}\left(\sum_A x_A \frac{\partial x_A}{\partial q_i}\frac{\partial x_A}{\partial q_j}\right)\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j} T_{ij}(q)\ \dot{q}_i\dot{q}_j, \tag{6}$$

where the matrix $T_{ij}(q)$ depends on coordinates only. Thus, $T$ is still quadratic in velocity, but can also depend on coordinates.

(Next we discussed Purely Kinetic Lagrangians in GR [see Example Sheet 1, Problem 10].)

**Corollary 2.1** *For closed and scleronomous systems Hamiltonian equals total energy of the system.*

**Proof** From (6) it follows that $T$ is a homogeneous function of degree 2 in $\dot{q}$. Indeed, $T(\alpha\dot{q}) = \alpha^2 T(q)$, $\forall\ \alpha \in \mathbb{R}$.

By *Euler's Theorem for homogenous functions*

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T.$$

Thus,

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - L = T + V = E.$$

□

*Case 2 - Rheonomic constraints:* $x_A = x_A(\mathbf{q}, \mathbf{t})$ and therefore

$$\sum_A \dot{x}_A^2 = \frac{1}{2}\sum_{i,j}\left(\sum_A x_A \frac{\partial x_A}{\partial q_i}\frac{\partial x_A}{\partial q_j}\right)\dot{q}_i\dot{q}_j + \frac{1}{2}\sum_i\left(\sum_A x_A \frac{\partial x_A}{\partial q_i}\frac{\partial x_A}{\partial t}\right)\dot{q}_i + \left(\frac{\partial x_A}{\partial t}\right)^2,$$

that is $T$ is not homogenius in $\dot{q}$ anymore. Lagrangian might still not depend on time explicitely, in which case Hamiltonial will be conserved. However, total energy will not be conserved (as in the example of the bead on rotating hoop - application II).

**Small Oscillations, Stability and Normal Modes**
We considered equations of motion of a physical system with $n$ degrees of freedom

$$\ddot{q}_i = f_i(q_1, ..., q_n) = -\frac{\partial V(\mathbf{q})}{\partial q_i}, \quad \text{for } i = 1, ..., n.$$

If $\mathbf{q^0}$ is an equilibrium point, then $f_i(q_1^0, ..., q_n^0) = 0$ for all $i$. We cosidered small pertrubations away from the equilibrium: $q_i(t) = q_i^0 + \eta_i(t)$ and expanded equation of motion in linear order in $\eta$

$$\ddot{\eta}_i = \sum_j \left.\frac{\partial f_i}{\partial q_j}\right|_{\mathbf{q^0}} \eta_j,$$

which can be written in a matrix form as $\ddot{\boldsymbol{\eta}} = F\boldsymbol{\eta}$.