Lectures 11-14 (1st, 4rd, 6th and 8th of November 2014)

Comments and questions should be sent to Berry Groisman, bg268@.

3 Rotation of Rigid Bodies

3.1 Kinematics: Angular Velocity

A rigid body (RB) does not have any internal degrees of freedom. Its motion is fully specified by six degrees of freedom: 3 translational and 3 rotational. Let us consider rotational part of motion of a RB. We choose a point P (normally within the body) and describe the body's rotation about P. Two reference frames are of interest: the space frame $\{\tilde{\boldsymbol{e}}_a\}$ and a frame embedded within the body - the body frame $\{\boldsymbol{e}_a(t)\}$. The space frame is fixed, while the body frame rotates with respect to the space frame. These two frames are assumed being orthonormal and are related by a 3×3 matrix R_{ab} as follows

$$\mathbf{e}_a(t) = R_{ba}(t)\ \tilde{\mathbf{e}}_b, \tag{1}$$

where R_{ba} is an orthogonal matrix. Its orthogonality follows from orthonormality of the bases. Indeed,

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab} = R_{ca}R_{db}\ \tilde{\boldsymbol{e}}_c \cdot \tilde{\boldsymbol{e}}_d = R_{ca}R_{db}\delta_{cd} \quad \Rightarrow \quad R_{ca}R_{cb} = \delta_{ab}. \tag{2}$$

R is unique - it follows by construction.
In other words, rotation of a RB is described by a t-dependent orthogonal matrix. Recall that orthogonal matrices have determinant ± 1. Rotation matrix corresponds to $\det R = 1$. Three parameters are needed to specify such rotation: the total number of components of 3×3 is 9, but we subtract 6 - the number of parameters which are fixed by orthogonality relations. Thus, three generalised coordinated are needed to parameterise the configuration space $\mathcal{C}$. In other words, to specify orientation of $\{\boldsymbol{e}_a\}$ with respect to $\{\tilde{\boldsymbol{e}}_a\}$ three numbers are required.

Positions vector $\boldsymbol{r}$ of any point within the body can be expanded in either the space frame or the body frame as follows

$$\begin{aligned} \boldsymbol{r}(t) &= \tilde{r}_a(t)\tilde{\boldsymbol{e}}_a \quad \text{in the space frame} \\ &= r_a\boldsymbol{e}_a(t) \quad \text{in the body frame.} \end{aligned} \tag{3}$$

(Notice that in the former case these are the components which depend on time, while in the latter these are the coordinate axes.) We can derive the relation between the components of the vector in the two frames by substituting (1) into the second line:

$$\boldsymbol{r}(t) = \boldsymbol{r}_a R_{ba}\tilde{\mathbf{e}}_b \quad \Rightarrow \quad \tilde{r}_a(t) = r_a R_{ba}(t). \tag{4}$$

Lets us see how the body frame axes change with time.

$$\frac{d\boldsymbol{e}_a(t)}{dt} = \frac{dR_{ba}(t)}{dt}\tilde{\boldsymbol{e}}_b = \frac{dR_{ba}(t)}{dt}\left(R^{-1}\right)_{cb}\boldsymbol{e}_c = \omega_{ac}\boldsymbol{e}_c, \tag{5}$$

where we introduce a new matrix

$$\omega_{ac} \equiv \dot{R}_{ba} \left(R^{-1}\right)_{cb} = \dot{R}_{ba} R_{bc}. \tag{6}$$

Here we used the fact that for orthogonal matrices $R^{-1} = R^T$.
Orthogonality of R implies that ω is anti-symmetric. Indeed,

$$R_{ba} R_{bc} = \delta_{ac} \quad \Rightarrow \quad \dot{R}_{ba} R_{bc} + R_{ba} \dot{R}_{bc} = \omega_{ac} + \omega_{ca} = 0 \quad \Rightarrow \quad \omega_{ac} = -\omega_{ca}. \tag{7}$$

Therefore, ω_{ac} has only three real parameters as entries

$$\omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}. \tag{8}$$

This justifies defining an object with a single index, ω_b as follows.

$$\omega_b = \frac{1}{2} \epsilon_{bac} \omega_{ac} \quad \Rightarrow \quad \omega_{ac} = \epsilon_{acb} \omega_b \tag{9}$$

and

$$\omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{10}$$

Entries ω_b are treated as components of a "vector", in the body frame, $\boldsymbol{\omega} = \omega_a \boldsymbol{e}_a$. Vector $\boldsymbol{\omega}$ is not a real vector, but *pseudo-vector* or *axial* vector. It does not define a direction to a specific point. This is a vector of *instantaneous angular velocity* with components measured with respect to the body frame.

Recall equation (5)

$$\frac{d\boldsymbol{e}_a(t)}{dt} = \omega_{ac} \boldsymbol{e}_c = \epsilon_{acb} \omega_b \boldsymbol{e}_c = -\epsilon_{abc} \omega_b \boldsymbol{e}_c = -\boldsymbol{e}_a \times (\omega_b \boldsymbol{e}_b) = \boldsymbol{\omega} \times \boldsymbol{e}_a, \tag{11}$$

where we used the fact that $\{\boldsymbol{e}_a\}$ form a right-hand coordinate system, $\epsilon_{abc} \boldsymbol{e}_c = \boldsymbol{e}_a \times \boldsymbol{e}_b$.
Now, from (3) we obtain

$$\boldsymbol{r}(t) = r_a \boldsymbol{e}_a(t) \quad \Rightarrow \quad \dot{\boldsymbol{r}}(t) = r_a \dot{\boldsymbol{e}}_a(t) = \boldsymbol{\omega} \times (r_a \boldsymbol{e}_a)(t) = \boldsymbol{\omega} \times \mathbf{r}(t). \tag{12}$$

The latter result should be familiar to some and could have been derived from simple geometrical considerations.

Thus, using matrix R we derived instantaneous angular velocity with components in the body frame. A different type of angular velocity can be defined in relation to the space frame, but we leave it beyond the scope of this course.

3.2 Tensor of Inertia, Kinetic Energy and Angular Momentum

The expression for kinetic energy of a system of N particles $T = 1/2 \sum_i m_i \dot{\boldsymbol{r}}_i{}^2$ can be easily generalized to a continuous body. Using the results obtained in the previous section we write

$$\begin{aligned} T &= \frac{1}{2} \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \dot{\boldsymbol{r}}^2 \\ &= \frac{1}{2} \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) = \frac{1}{2} \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\boldsymbol{r} \cdot \boldsymbol{r}) - (\boldsymbol{r} \cdot \boldsymbol{\omega})(\boldsymbol{r} \cdot \boldsymbol{\omega})\right] \\ &= \frac{1}{2} \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \left[(\boldsymbol{r} \cdot \boldsymbol{r}) \omega_a \omega_b \delta_{ab} - (r_a \omega_a)(r_b \omega_b)\right] = \frac{1}{2} \omega_a \omega_b \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \left[(\boldsymbol{r} \cdot \boldsymbol{r}) \delta_{ab} - (r_a r_b)\right], \end{aligned} \tag{1}$$

where the components of position vectors and the angular velocity are given in the *body frame*. We will assume that the body is rigid. The quantity given by the integral in the RHS is known as the *Tensor of Inertia*.

$$I_{ab} = \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \left[(\boldsymbol{r} \cdot \boldsymbol{r}) \delta_{ab} - (r_a r_b)\right]. \tag{2}$$

Thus, T can be rewritten in a compact form as

$$T = \frac{1}{2} I_{ab} \omega_a \omega_b = \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega}. \tag{3}$$

Key properties of I:

(1) It is measured in the body frame and is independent of time

(2) It is calculated with respect to a particular point - the origin of the body frame

(3) It is symmetric: $I_{ab} = I_{ba}$

(4) It has real elements (from which is follows that)

(5) It is diagonalisable.

Let us expand on the last property. It implies that there exists an orthogonal matrix O such that OIO^T is a diagonal matrix. In terms of the body frame diagonalisation means that we rotate $\{\boldsymbol{e}_a\}$ until the axes coincide with the eigenvectors of I. The body frame basis axes in which I is diagonal are called *Principal Axes* and the corresponding eigenvalues of I are called *Principal Moments of Inertia*, ${I_a}$[1]. It will be normally assumed that $\{\boldsymbol{e}_a\}$ are principle axes. Then kinetic energy takes a very simple and convenient form

$$T = \frac{1}{2} I_a \omega_a^2. \tag{4}$$

The kinematical properties of a rigid body are fully determined by its mass, principal axes and moments of inertia.
Examples: (we will look at couple of simple examples, time permitting.)

Theorem 3.2.1 Parallel Axis Theorem: *Consider the tensor of inertia, $(I_{CoM})_{ab}$, measured w.r.t. the centre of mass of the body of mass M. The Tensor of Inertia w.r.t. a point P' displaced from the centre of mass by constant vector $\boldsymbol{c}$ is*

$$(I_{P'})_{ab} = (I_{CoM})_{ab} + M \left[(\boldsymbol{c} \cdot \boldsymbol{c}) \delta_{ab} - (c_a c_b) \right]. \tag{5}$$

Proof To prove start with the explicit expression

$$(I_{P'})_{ab} = \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \left[(\boldsymbol{r} - \boldsymbol{c}) \cdot (\boldsymbol{r} - \boldsymbol{c}) \delta_{ab} - (\boldsymbol{r} - \boldsymbol{c})_a (\boldsymbol{r} - \boldsymbol{c})_b) \right], \tag{6}$$

and use the definition of the CoM,

$$\int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \boldsymbol{r} = 0. \tag{7}$$

□

Angular Momentum: angular momentum is familiar, but now it is time to relate it to I_{ab} and $\boldsymbol{\omega}$.

$$\boldsymbol{L} = \int d^3\boldsymbol{r} \, (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = \int d^3\boldsymbol{r} \, (\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})) = \int d^3\boldsymbol{r} \rho(\boldsymbol{r}) \left[(\boldsymbol{r} \cdot \boldsymbol{r}) \, \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{r}) \, \boldsymbol{r} \right] = I \boldsymbol{\omega}. \tag{8}$$

Thus, we write the components of $\boldsymbol{L}$ in the body frame $\boldsymbol{L} = L_a \boldsymbol{e}_a$, where

$$L_a = I_{ab} \omega_b. \tag{9}$$

It is important to emphasize that our intuition often leads us to believe that angular velocity and angular momentum are parallel. However, this is not true in general. As we will see below for many rotating bodies $\boldsymbol{L} \neq k\boldsymbol{\omega}$, $k \in \mathbb{R}$ (with both vectors generally rotating with respect to the body frame).

Until now we discussed rotation of RB about a point fixed wrt to the body itself. Let us now consider the overall motion of the body with respect to a fixed space frame $\{\tilde{\boldsymbol{e}}_a\}$. Earlier we have shown that the overall motion of the body can be split into translational motion of the CoM and rotation about the CoM, thus total kinetic energy can be written as

[1] The Principal Moments, I_a, are measured about a particular axis of rotation, hence familiar simple definition $I = \int \rho(r) r_\perp^2 dV$. In the third example sheet you will be asked to prove that $I_a \in \mathbb{R}$ and $I_a \geq 0$.

$$T = \frac{M\dot{\boldsymbol{R}}^2}{2} + \frac{1}{2}\int d^3\boldsymbol{r}\rho(\tilde{\boldsymbol{r}})\dot{\tilde{\boldsymbol{r}}}^2 = \frac{M\dot{\boldsymbol{R}}^2}{2} + \frac{1}{2}I_{ab}^{\mathrm{CoM}}\omega_a\omega_b, \tag{10}$$

where $\boldsymbol{R}$ is the position vector of the CoM in the space frame. In the space frame, the Tensor of Inertia *wrt to the CoM* will be written as

$$I_{ab}^{\mathrm{CoM}} = \int d^3\boldsymbol{r}\rho(\boldsymbol{r})\left[(\boldsymbol{r}-\boldsymbol{R})\cdot(\boldsymbol{r}-\boldsymbol{R})\delta_{ab} - (\boldsymbol{r}-\boldsymbol{R})_a(\boldsymbol{r}-\boldsymbol{R})_b)\right]. \tag{11}$$

3.3 Euler's Equations

Euler's Equations are relations between principle moments of inertia, I_a, and the components of the angular velocity, $\boldsymbol{\omega}$, in the body frame. Recall that

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau}^{\mathrm{ext}}. \tag{1}$$

In the body frame $\boldsymbol{L} = L_a\boldsymbol{e}_a$ and so

$$\frac{d\boldsymbol{L}}{dt} = \frac{dL_a}{dt}\boldsymbol{e}_a + L_a\frac{d\boldsymbol{e}_a}{dt} = \frac{dL_a}{dt}\boldsymbol{e}_a + L_a(\boldsymbol{\omega}\times\boldsymbol{e}_a) = \boldsymbol{\tau}^{\mathrm{ext}}. \tag{2}$$

Also,if $\{\boldsymbol{e}_a\}$ coincide with the Principal Axes then $L_a = I_a\omega_a$. We rewrite (2) in components and use the above result for L_a. Thus, we derive Euler's Equations for the components of angular velocity $\boldsymbol{\omega}$ in body frame

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= \tau_1^{\mathrm{ext}}, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= \tau_2^{\mathrm{ext}}, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= \tau_3^{\mathrm{ext}}, \end{aligned} \tag{3}$$

where τ_a^{ext} are components of external torque.

3.4 Free Tops

Free tops are bodies which are not subject to torques, i.e. free rotating bodies.
We discuss the following examples of free tops.

3.4.1 The Symmetric Top

When $I_1 = I_2 = I_3$, such as a sphere, the motions is simple - $\boldsymbol{\omega}$ is conserved and the body spins around a fixed axis with constant angular frequency. In this case $\boldsymbol{L}$ is parallel to $\boldsymbol{\omega}$.

When $I_1 = I_2 \neq I_3$, e.g. in case of the Symmetric Ellipsoid, the motion is more interesting. Using Euler's Equations (3) we show that ω_3 component is conserved, while the vector $\boldsymbol{\omega}$ precessed in $\boldsymbol{e}_1, \boldsymbol{e}_2$-plane with angular frequency

$$\Omega = \omega_3\frac{I_3 - I_1}{I_1}.$$

More explicitly, we rewrite the first two Euler's equations as

$$\begin{cases} \dot{\omega}_1 = -\Omega\omega_2, \\ \dot{\omega}_2 = +\Omega\omega_1. \end{cases} \tag{1}$$

The two components can now be decoupled

$$\begin{cases} \ddot{\omega}_1 + \Omega^2\omega_1 = 0, \\ \ddot{\omega}_2 + \Omega^2\omega_2 = 0, \end{cases} \tag{2}$$

which obviously yields the solution

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \omega_0\begin{pmatrix} \sin(\Omega t + \phi_0) \\ \cos(\Omega t + \phi_0) \end{pmatrix}.$$

In the space frame the motion looks like a wobble. It is a combination of two precessions: one of $\boldsymbol{\omega}$ about $\boldsymbol{e}_3$ and the other of $\boldsymbol{e}_3$ about $\boldsymbol{L}$ $(\tilde{e}_3)$. Notice that $\boldsymbol{\omega}$ and $\boldsymbol{L}$ are not parallel!

3.4.2 General Asymmetric Top

Poinsot Construction

First, we analyse a simplest case of rotation of a top about one of its principal axes. We discussed stability of such rotations and we have shown that rotation is stable about the axis with the largest of smallest I.

Next, we discuss Poinsot construction, named after French mathematician Louis Poinsot. This is a nice geometrical representation of the motion of $\boldsymbol{\omega}$ or $\boldsymbol{L}$ in the body frame. We start with writing the two constants of motion in this case, namely

$$
\begin{aligned}
T &= \frac{I_1\omega_1^2}{2} + \frac{I_2\omega_2^2}{2} + \frac{I_3\omega_3^2}{2}, \\
\boldsymbol{L}^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2.
\end{aligned}
$$

They can be rewritten as two ellipsoids with axes ω_a ($\boldsymbol{\omega}$-space representation). The motion of the vector $\boldsymbol{\omega}$ lies on the intersection of the two ellipsoids (the first is known as *inertia ellipsoid*).

Alternatively, we can rewrite them in terms of components L_a ($\boldsymbol{L}$-space representation). In this case the first is an ellipsoid and the second is a sphere with radius L. The motion of the vector $\boldsymbol{L}$ lies on the intersection of the ellipsoid and the sphere. We discuss in detail this latter possibility (including sketch on the board).

Solution in Terms of Elliptic Integrals

Now its time for the quantitative treatment. *The full derivation is quite long and technical, so I have prepared a separate handout. I am asking you to read it first - I will then give a general overview in class and answer any questions.*