Lectures 3 and 4 (14th, 16th of October 2014)

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2 The Lagrangian Formalism

2.1 Configuration Space

Key concepts: configuration space $\mathcal{C}$, degrees of freedom, constraints, Lagrangian

Consider a mechanical system of N particles. Their positions are specified by N vectors, $\{\boldsymbol{r}_i\}_{i=1}^{N}$, in 3-dimensional Euclidean space. We rewrite these vectors as a *single* vector in $3N$-dimensional *configuration space*, $\mathcal{C}$, $\mathbf{x} = (x_1, x_2, ..., x_k, ..., x_{3N})$. Now, the position of all particles is specified by a single point in $\mathcal{C}$ and the dynamical evolution of a system is represented by a single trajectory in $\mathcal{C}$.

For many systems dimensionality of $\mathcal{C}$ can be reduced due to constraints, thus the number of *degrees of freedom* is often less than $3N$ (e.g. solid bodies do not require $\sim 10^{23}$ dimensions). In such cases we will introduce a different set of coordinates, *generalized coordinates*, which do not have to correspond directly to real coordinates in Euclidean Space. We will return to generalized coordinates in Section 2.4. For now, let us assume that $\mathcal{C}$ is an $3N$-dimensional direct-product of N real spaces, $\mathbb{R}^3$, of individual particles, as defined above.

Postulate 2.1 *The system is fully characterised by the Lagrangian function, $L(\mathbf{x}, \dot{\mathbf{x}}, t)$, the form of which will be specified in the next section.*

2.2 The Principle of Least Action

Key concepts: action, functional, variational calculus, Hamilton Principle, Euler-Lagrange equation, constraints

(Without loss of generality, in this section we will consider a single component x_k and omit the index k.)

Assume that at times t_1 and t_2 the system's position in $\mathcal{C}$ is fixed: $x(t_1) = x_1$ and $x(t_2) = x_2$. We consider all smooth paths $x(t)$ in $\mathcal{C}$ with these fixed points. To each path let us assign a number

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}, t)dt, \tag{1}$$

which is a functional (the action integral).

Postulate 2.2 *The Principle of Least Action (Hamilton Principle) states that the actual path taken by the system corresponds to a stationary value of S.*

Using the tools of Variational Calculus we deduce that the condition for extremum, $\delta S = 0$ is equivalent to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{2}$$

$3N$ Euler-Lagrange equation(s) (1 for each component of $\mathbf{x}$).

Derivation of E-L equation: Proof Assume that $x(t)$ is the actual path. Fix t and vary $x(t)$, $\dot{x}(t)$, i.e. $x(t) \to x(t) + \delta x(t)$. For a 'stationary value' of S small variations in $x(t)$ (once it's found) can produce only 2nd-order variations in the integral.

$$\delta S = S[x + \delta x] - S[x] = \int_{t_1}^{t_2} L(x + \delta x, \dot{x} + \delta \dot{x}, t)dt - \int_{t_1}^{t_2} L(x, \dot{x}, t)dt.$$

We expand L in Taylor series in two variables (δx, $\delta \dot{x}$) at a fixed value of t:

$$L(x + \delta x, \dot{x} + \delta \dot{x}, t)dt = L(x, \dot{x}, t) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \mathcal{O}(\delta x^2, ...).$$

Thus, in the limit $\delta x \to 0$, $\delta \dot{x} \to 0$

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right) dt,$$

up to the second order. The second term of the integrand is calculated by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}}\delta \dot{x} dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \left(\frac{d}{dt}\delta x \right) dt = \left[\frac{\partial L}{\partial \dot{x}}\delta x \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \delta x dt = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \delta x dt.$$

Thus,

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right] \delta x dt = 0.$$

As δx is arbitrary we deduce that the expression within the brackets must be zero.

□

Let us review several main properties of the Lagrangian, in particular:

Property 1: Lagrangian is defined up to a full derivative of an arbitrary function of coordinates and time.

Consider $L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{d}{dt}f(x, t)$.

$$S' = S + \int_{t_1}^{t_2} \frac{d}{dt}f(x, t)dt = S + f(x_2, t_2) - f(x_1, t_1).$$

Hence, $\delta S' = \delta S$, that is adding f does not change the equations of motion.

Property 2: (form of the Lagrangian): For a system of particles $L = T - V$, where T and V are total kinetic and potential energies respectively: $T = 1/2 \sum_k m_k \dot{x}_k^2$, $V = V(\boldsymbol{x})$, where the mass m_k is traced back to a corresponding particle.

E-L eqns take the form (for each component): $\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$, so

$$\dot{p} = -\frac{\partial V}{\partial x} \qquad \text{Newton's Equation}$$

Property 3: Since T can be made arbitrary large, S is not bounded from above. It can be a minimum or a saddle point.

Let us also discuss three benefits of using the Lagrangian:

Benefit 1: Unlike Newton's equation, E.-L. equation holds in *any* reference frame (coordinate system).

Benefit 2: It is easier to deal with constraints in Lagrangian formalism.

Benefit 2: All fundamental laws of physics (and even beyond) can be written in terms of the action principle.

Let us discuss these benefits in detail.

2.3 Changing coordinate systems

Here we show that E-L equation holds in *any* coordinate system.

The Principle of Least Action is a statement about paths, not coordinates, so intuitively we should expect that E-L equations will hold in any coordinate system. We can prove this statement explicitly.

We introduce new coordinates $q_a = q_a(\mathbf{x}, t)$ (which could be t-dependent) and prove that if E-L equation holds in x-system, then it holds in q-system, i.e.

Statement 2.3

$$\text{If} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k} = 0 \quad \text{then} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0.$$

Proof Let us invert the coordinate relationship, which is possible under condition $\det\left(\partial x_k/\partial q_a\right) \neq 0$,

$$x_k = x_k(\boldsymbol{q}, t),$$

and calculate the velocity

$$\dot{x}_k = \frac{\partial x_k}{\partial q_a}\dot{q}_a + \frac{\partial x_k}{\partial t}, \tag{3}$$

where Einstein summation convention is used.

We then substitute $x_k(\boldsymbol{q}, t)$ and $\dot{x}_k(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ into $L(x_k, \dot{x}_k, t)$ and calculate its derivatives:

$$\frac{\partial L}{\partial q_a} = \frac{\partial L}{\partial x_k}\frac{\partial x_k}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_k}\frac{\partial \dot{x}_k}{\partial q_a} = \frac{\partial L}{\partial x_k}\frac{\partial x_k}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_k}\left(\frac{\partial^2 x_k}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x_k}{\partial t \partial q_a}\right). \tag{4}$$

$$\frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial x_k}\frac{\partial x_k}{\partial \dot{q}_a} + \frac{\partial L}{\partial \dot{x}_k}\frac{\partial \dot{x}_k}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{x}_k}\frac{\partial x_k}{\partial q_a}, \tag{5}$$

where we used $\partial x_k/\partial \dot{q}_a = 0$ and $\partial \dot{x}_k/\partial \dot{q}_a = \partial x_k/\partial q_a$. The latter is a corollary of (3). Thus,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right)\frac{\partial x_k}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{dt}\left(\frac{\partial x_k}{\partial q_a}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right)\frac{\partial x_k}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_k}\left(\frac{\partial^2 x_k}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x_k}{\partial t \partial q_a}\right). \tag{6}$$

From (4) and (6) it follows that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k}\right]\frac{\partial x_k}{\partial q_a}.$$

Recall that for invertible transformation the partial derivative in the RHS is non-zero and deduce that if E-L eqn is satisfied in x-coordinate system, then it is satisfied in q-coordinate system.

□

Example: Consider a free particle moving with velocity $\dot{\boldsymbol{r}}$ (in Cartesian coordinates $\boldsymbol{r} = (x, y, z)$). The Lagrangian of a free particle in the inertial frame is $L = m\dot{\boldsymbol{r}}^2/2$. We introduced a coordinate system, which rotates with velocity $\boldsymbol{\omega} = (0, 0, \omega)$ about z-axis and show that L takes the form

$$L = \frac{m}{2}(\dot{\boldsymbol{r}}' + \boldsymbol{\omega} \times \boldsymbol{r}')^2.$$

From E-L equations we then derive equations of motion

$$\ddot{\boldsymbol{r}}' = \boldsymbol{0} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') - 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}',$$

where the three terms in the RHS are associated with real force, centrifugal and Coriolis forces.