Lectures 3, 4 and 5 (8th, 11th, 13th of October 2012) - outline

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2 The Lagrangian Formalism

2.1 The Principle of Least Action

Key concepts: configuration space $\mathcal{C}$, degrees of freedom, action, functional, variarional calculus, Euler-Lagrange equation, constraints (holonomic: rheonomic and scleronomic)

Consider a system of N particles and introduce a vector in *configuration space*, $\mathcal{C}$, $\mathbf{x} = (x_1, x_2, ..., x_{3N})$.

The system is characterised by a function $L(\mathbf{x}, \dot{\mathbf{x}}, t)$, the Lagrangian.
(Considering a single component) assume that at times t_1 and t_2 the system's position in $\mathcal{C}$ is fixed: $x(t_1) = x_1$ and $x(t_2) = x_2$. We condider all smooth paths $x(t)$ in $\mathcal{C}$ with these fixed points. To each path let us assign a number

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}, t)dt, \tag{1}$$

which is a functional (the action integral).
The Principle of Least Action (Hamilton Principle) states that the actual path taken by the system is an extrememum of S.

Using the tools of Variational Calculus we deduce that the condition for extremum, $\delta S = 0$ is equivalent to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{2}$$

$3N$ Euler-Lagrange equation(s) (1 for each component of $\mathbf{x}$).

We reviewed several properties of the Lagrangian, in particular:

Property 1: Unlike Newton's equation, E.-L. equation holds in *any* reference frame.

Property 2: It is easier to deal with constraints in Lagrangian formalism.

Property 3: Lagrangial is defined up to a full derivative of an arbitrary function of coordinates and time.

Property 4 (form of the Lagrangian): $L = T - V$, where T and V are total kinetic and potential energies respectively.

Property 6: Since T can be made arbitrary large, S is not bounded from above. It can be a minimum or a saddle point.

2.2 Changing coordinate systems

We introduced new coordinates $q_a = q_a(\mathbf{x}, t)$ and proved that if E-L equation holds in x-system, then it holds in q-system, i.e.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0. \tag{7}$$

We gave an example of a free particle moving with velocity $\dot{\mathbf{r}}$ (in Cartesian coordinates). The Lagrangian of a free particle in the inertial frame is $L = m\dot{\mathbf{r}}^2/2$. We introduced a coordinate system, which rotates with velocity $\boldsymbol{\omega} = (0, 0, \omega)$ about z-axis and showed that L takes the form

$$L = \frac{m}{2}(\dot{\mathbf{r}}' + \omega \times \mathbf{r}')^2.$$

From E-L equations we then derived equations of motion

$$\ddot{\mathbf{r}}' = \mathbf{0} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}',$$

where the three terms in the RHS are associated with real force, centrifugal and Coriolis forces.

2.3 Constraints and Generalised Coordinates

We started with an example of a Simple Mathematical Pendulum to demonstrate how we deal with constraints in Newtonian formalism. We noted that a lot of effort was put into dealing with the constraint force (tension of the string) there. Lagrangian formalism allows us to avoid the extra work associated with the contraint force(s), which is useful if we are not interested in them.

Holonomic constraints are relationaships between the coordinates of the form

$$f_\alpha(\mathbf{x}, t) = 0,$$

where $\alpha = 1, ..., 3N - n$ being a number of constraints (with some $n < 3N$). H.C. can be resolved in terms of n *generalised coordinates*, q_i, with $i = 1, 2, ..., n$, so $\mathbf{x} = \mathbf{x}(\mathbf{q})$ (which contains constraint implicitely). The system is said to have n *degrees of freedom*.
If f_α depends on t explicitely then the constraint is called *rheonomic*. It is called *scleronomic* otherwise.
We gave 3 examples of scleronomic constraints.

Difficulties in incorporting constraints into Lagrangian Formalism:
Recall the action principle (written with explicit summation over all components of $\mathbf{x}$)

$$\delta S = \int_{t_1}^{t_2} \left[\sum_A \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_A} - \frac{\partial L}{\partial x_A} \right) \delta x_A \right] dt = 0.$$

To conclude that E-L equation is satisfied for each component independently we used an implicit assumption, that all components of $\mathbf{x}$ are independent and so δx_A can be varied independently. However, when constraints are imposed we cannot conclude that each coefficient of the variations δx_A vanishes by itself and so there is no justification for writing $3N$ independent E-L equations.