Lecture 5 and 6 (18th and 21st of October 2014)

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2.4 Constraints and Generalised Coordinates

Key concepts: constraints (holonomic: rheonomic and scleronomic), generalized coordinates, Lagrange multipliers, generalised velocities and momenta, conjugate quantities.

So far we have considered systems of N particles. Such systems have $3N$ degrees of freedom and hence require $3N$ equations of motion to determine their dynamics. Many physical systems, however, move under constraints. The typical example is a rigid body. Under reasonable approximation the constraints are expressed in the form

$$f_{ij}(\mathbf{r}_i, \mathbf{r}_j) = (\mathbf{r}_i - \mathbf{r}_j)^2 - a_{ij}^2 = 0, \tag{1}$$

where a_{ij} are fixed distanced between the corresponding pairs of particles. These constraints are written in terms of particles' positions in real space, but they can be stated in terms of coordinates of $\mathbf{x}$ in $\mathcal{C}$. Expression (1) is a typical example of what is known as *holonomic* constraints.

Holonomic Constraints (HC) are relationships between the coordinates of the form

$$f_\alpha(\mathbf{x}, t) = 0, \tag{2}$$

where $\alpha = 1, ..., 3N - n$ is the number of constraints and $0 < n < 3N$.

HC can be solved in terms of n *generalized coordinates* q_i, $i = 1, ..., n$, so the old coordinates $\mathbf{x} = \mathbf{x}(\mathbf{q})$ contain the constraints implicitly. The system is said to have *n degrees of freedom.* If (2) do not contain time explicitly then the HC is called *scleronomic* and the system is said to be *scleronomous.* If (2) does contain t explicitely then the HC is called *rheonomic* and the system is said to be *rheonomous.*

Examples: Three examples of HC have been discussed in detail during the lecture:

(1) Planar Pendulum: motion on a circle (2 constraints, 1 degree of freedom (DofF))

(2) Spherical Pendulum: motion on the surface of a sphere (1 constraint, 2 DofF)

(3) Bead sliding on a flat wire (2 constraints, 1 DofF). If radii or the circle/sphere or the shape of the wire are fixed, then the constraint is scleronomic. Constrains before rheonomic if these quantities change with time.

Difficulties in incorporating constraints into Newtonian Formalism:
Constraints introduce two types of difficulties in the solution of mechanical problems. In order to illustrate the idea consider the equation of motion of a constrained system of N particles written in Newtonian form - see Equations (9) and (10) in Section 1.2 (for convenience lets work in real space). The problem constitutes in solving the set of differential equations for coordinates

$$m_i \ddot{\mathbf{r}}_i = \sum_{j \neq i} \boldsymbol{F}_{ij} + \boldsymbol{F}_i^{\text{ext}}. \tag{3}$$

First difficulty: due to the constraints the coordinates r_i are no longer all independent. Second difficulty: Constraint forces are among the unknowns of the problem. They are not know a priori and known rather in terms of their effect on the motion of the system.

Using a simple example of a planar pendulum (with mass m and length l) we illustrate how these difficulties are dealt with in the framework of Newtonian formalism.

We write the Newton's equations of motion

$$\begin{cases} m\ddot{x} = mg - T\cos\theta, \\ m\ddot{y} = -T\sin\theta, \end{cases}$$

and then impose the constaint $x^2 + y^2 = l^2$ which is parametrized by

$$\begin{cases} x = l\cos\theta, \\ y = l\sin\theta. \end{cases}$$

Once the contraint is incorporated into the Newton's equations we (eventually) obtain

$$\begin{cases} \ddot{\theta} = -\dfrac{g}{l}\sin\theta, \\ T = ml\dot{\theta}^2 + mg\cos\theta. \end{cases}$$

It is evident that a considerable effort has been put into dealing with the constraint force (tension of the string, T). Lagrangian formalism allows us to avoid the extra work associated with the contraint force(s), which is useful if we are not interested in them.

Difficulties in incorporating constraints into Lagrangian Formalism:
Recall the action principle (written with explicit summation over all components of $\mathbf{x}$)

$$\delta S = \int_{t_1}^{t_2} \left[\sum_k \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k} \right) \delta x_k \right] dt = 0.$$

To conclude that E-L equation is satisfied for each component independently we used an implicit assumption, that all components of $\mathbf{x}$ are independent and so δx_k can be varied independently. However, when constraints are imposed we cannot conclude that each coefficient of the variations δx_k vanishes by itself and so there is no justification for writing $3N$ independent E-L equations.

During the lecture in the class we show how constraints can be incorporated into Lagrangian formalism with the help of Lagrange multipliers. We introduce modified Lagrangian

$$L' = L(\mathbf{x}, \dot{\mathbf{x}},) + \lambda_\alpha f_\alpha$$

and obtain the following modification of E-L. equations, where the term in the RHS is a manifistation of the contraints:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} - \frac{\partial L}{\partial x_k} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x_k}. \tag{8}$$

Thus, we demonstrate that we can easily incorporate constraint forces into the Lagrangian set-up using Lagrange multipliers.

In many situations, however, we might not be interested in contraint forces, but just in the dynamics of the generalised coordinates, q_i. We prove the following statement.

Theorem 2.1 *For constrained systems we may derive the equations of motion directly in generalised coordinates, i.e. using*

$$L(q_i, \dot{q}_i, t) = L\left(x_k(q_i, t), \dot{x}_k(q_i, \dot{q}_i, t)\right)$$

and E-L equations written purely in terms of generalised coordinates.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

In other words, we may ignore λ_α and work with unconstrained Lagrangian where we substitute $x_k = x_k(\mathbf{q}, t)$.

Summary: General form of the Lagrangian Formalism

A mechanical system is described by n generalised coordinates (degrees of freedom), q_i, which define a point in a n-dimensional configuration space $\mathcal{C}$.
The time evolution is a curve in $\mathcal{C}$ governed by the Lagrangian $L(q_i, \dot{q}_i, t)$, such that q_i obey n coupled 2nd order differential equations (Euler-Lagrange equations)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

Some definitions:

- $\dot{q}_i$ - generalised velocities.
- $p_i = \frac{\partial L}{\partial \dot{q}_i}$ - generalised momenta conjugate to q_i.
- E-L eqns can be rewritten as

$$\dot{p}_i = \frac{\partial L}{\partial q_i}.$$

A note: sometimes $\dot{q}_i$ and p_i coincide with real velocities and momenta in Cartesian coordinates, however generally this is not the case. Consider, for example, polar coordinates $\dot{\mathbf{r}} = (\dot{r}, r\dot{\theta})$. For the planar pendulum $\dot{\mathbf{r}} = (0, l\dot{\theta})$, but it is natural to choose generalised velocity as $q = \dot{\theta}$ (not $l\dot{\theta}$), which does not have a dimension of L/T. In this case generalised momentum is $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, while the real momentum is $ml\dot{\theta}$.

2.5 Noether's Theorem and Symmetries

This sections discusses appearence of conservations laws in Lagrangian formalism. In particular, we state and prove Noether's Theorem, which relates conserved quantities to symmetries.

Definition: We say that $F(q_i, \dot{q}_i, t)$ is a constant of motion (conserved quantity) if

$$\frac{dF}{dt} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i\right) + \frac{\partial F}{\partial t} = 0,$$

where q_i satisfies E-L equations.
F remains constant along the path followed by the system.

We give two examples:

- If $\frac{\partial L}{\partial t} = 0$ then $H(q_i, p_i) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ (the Hamiltonian) is conserved.
- If $\exists\ q_j$ s.t. $\frac{\partial L}{\partial q_j} = 0$ then p_j is conserved. Such a coordinate is called *ignorable* or *cyclic.*