Lectures 7 and 8 (23rd and 25th of October 2014)

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Section 2.6 of these notes is likely to be updated after the lecture 7.

A note on notation: Einstein summation convention will be used throughout this course, however, in some places I will put an emphasis on summation by explicitly writing the sign.

2.5 Noether's Theorem and Symmetries

Key concepts: Noether's Theorem, symmetry.

This sections discusses appearance of conservations laws in Lagrangian formalism. In particular, we state and prove Noether's Theorem, which relates conserved quantities to symmetries.

Definition 2.5.1 *We say that $F(q_i, \dot{q}_i, t)$ is a constant of motion (conserved quantity) if*

$$\frac{dF}{dt} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial F}{\partial t} = 0, \tag{1}$$

where q_i satisfies E-L equations, i.e. F remains constant along the path followed by the system.

We give two examples:

- If $\frac{\partial L}{\partial t} = 0$ then $H(q_i, p_i) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ (the Hamiltonian) is conserved, which can be easily verified.
- If $\exists\ q_j$ s.t. $\frac{\partial L}{\partial q_j} = 0$ then p_j is conserved. Such a coordinate is called *ignorable* or *cyclic*.

Noether Theorem (NT) - proved by Emmy Noether in 1918:
Consider continuous one-parameter[1] family of transformations for the coordinates of the system

$$q_i(t) \to Q_i(s, t), \tag{2}$$

where $s \in \mathbb{R}$ (continuous parameter of the transformation), such that $Q_i(0, t) = q_i(t)$.

If L is invariant under this transformation, i.e.

$$L(Q_i(s, t), \dot{Q}_i(s, t), t) = L(q_i, \dot{q}_i, t), \tag{3}$$

then it must not depend on s:

$$\frac{d}{ds} L(Q_i(s, t), \dot{Q}_i(s, t), t) = 0. \tag{4}$$

If this is the case, then the transformation is said to be a *continuous symmetry of L.*

Theorem 2.5.2 *NT states that for each such symmetry there exists a conserved quantity.*

Proof (The proof constitutes in deriving such a quantity and providing the recipe for its calculation.)

$$\begin{aligned} \frac{dL}{ds} &= \frac{\partial L}{\partial Q_i} \frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i} \frac{d\dot{Q}_i}{ds} \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{Q}_i} \right) \frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i} \frac{d\dot{Q}_i}{ds}. \quad \text{(using E-L equations)} \end{aligned} \tag{5}$$

[1] Generalisation to several parameters is trivial.

Thus,

$$\left.\frac{dL}{ds}\right|_{s=0} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\left.\frac{dQ_i}{ds}\right|_{s=0}\right) = 0. \tag{6}$$

Thus, there is a conserved quantity, which is

$$\sum_i p_i \left.\frac{dQ_i}{ds}\right|_{s=0}. \tag{7}$$

□

In other words, NT provides the recipe: the conserved quantity can be found by differentiating each coordinate with respect to the parameters of the transformation in the immediate neighbourhood of the identity transformation, multiplied by corresponding generalised momentum and summing over all degrees of freedom.

Let us apply NT to the symmetries corresponding to three elements of the Galilean Group. For this purpose we use a closed system of N interacting particles with potential forces. The Lagrangian of such system is

$$L = \frac{1}{2}\sum_i m_i \dot{\boldsymbol{r}}_i^2 - \sum_{i,j(i<j)} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|). \tag{8}$$

- Homogeneity of Space $\Longrightarrow$ Translational Invariance of L $\Longrightarrow$ Conservation of Total Linear Momentum

  Consider a translation of a system as a whole by an arbitrary vector $s\hat{\boldsymbol{n}}$. Thus, $\boldsymbol{r}_i \to \boldsymbol{r}_i + s\hat{\boldsymbol{n}}$, $\forall\ \hat{\boldsymbol{n}},\ s$. **Let us clarify notation in this part.** In Eqn. (7) the index i runs over degrees of freedom/components in $\mathcal{C}$. For Lagrangian (8) of N interacting particles it is convenient to work in a vector form in real space, in which case (7) can be rewritten (in a vector form) as

  $$\sum_i \boldsymbol{p}_i \cdot \left.\frac{d\boldsymbol{Q}_i}{ds}\right|_{s=0}, \tag{9}$$

  where summation index i is over *particles*. The scalar product of the two vectors takes care of summation over all three degrees of freedom for each particle.

  Translation of the system as a whole does not change equations of motion, hence

  $$L(\boldsymbol{r}_i + s\,\hat{\boldsymbol{n}}, \dot{\boldsymbol{r}}_i, t) = L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, t). \tag{10}$$

  NT provided us with a recipe for computing the conserved quantity associated with this translation.

  $$\sum_i \boldsymbol{p}_i \cdot \left.\frac{d(\boldsymbol{r}_i + s\hat{\boldsymbol{n}})}{ds}\right|_{s=0} = \sum_i \boldsymbol{p}_i \cdot \hat{\boldsymbol{n}}, \tag{11}$$

  which is the total linear momentum along $\hat{\boldsymbol{n}}$. As this holds for any $\hat{\boldsymbol{n}}$, the total momentum, $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$, is conserved.

- Isotropy of Space $\Longrightarrow$ Rotational Invariance of L $\Longrightarrow$ Conservation of Total Angular Momentum

  Consider a rotation of a system as a whole around an arbitrary $\hat{\boldsymbol{n}}$. For an infinitesimal rotation the corresponding transformation of the coordinates is

  $$\boldsymbol{r}_i \to \boldsymbol{r}_i + \delta\boldsymbol{r}_i \approx \boldsymbol{r}_i + s\ (\hat{\boldsymbol{n}} \times \boldsymbol{r}_i), \tag{12}$$

  where s is assumed to be very small.

  Lagrangian of a closed system of N particles (Eqn. (8)) is invariant under such transformation, i.e.

$$L(\boldsymbol{r}_i + \delta\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i + \delta\dot{\boldsymbol{r}}_i, t) = L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, t), \tag{13}$$

that is in linear order

$$L(\boldsymbol{r}_i + s\,\hat{\boldsymbol{n}} \times \boldsymbol{r}_i, \dot{\boldsymbol{r}}_i + s\,\hat{\boldsymbol{n}} \times \dot{\boldsymbol{r}}_i, t) = L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i, t) + \left.\frac{dL}{ds}\right|_{s=0} s, \tag{14}$$

where the second term must be zero due to symmetry and the conserved quantity is calculated as

$$\sum_i \boldsymbol{p}_i \cdot \left.\frac{d}{ds}(\boldsymbol{r}_i + s\,\hat{\boldsymbol{n}} \times \boldsymbol{r}_i)\right|_{s=0} = \sum_i \boldsymbol{p}_i \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{r}_i) = \hat{\boldsymbol{n}} \cdot \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i, \tag{15}$$

which is total angular momentum in the direction of $\hat{\boldsymbol{n}}$. As $\hat{\boldsymbol{n}}$ is arbitrary we conclude that the total angular momentum is conserved.

- Homogeneity of Time: L is invariant under $t \to t + s$, that is $\frac{\partial L}{\partial t} = 0$, which implies (as we have shown earlier) that Hamiltonian

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L. \tag{16}$$

is conserved.
For a closed system (and more generally for systems with scleronomic constraints) Hamiltonian equals to the total energy, therefore we have the fundamental link

Homogeneity of Time $\Longrightarrow$ Conservation of Total Energy (for scleronomous systems)

It is worth noting that for some rheonomous systems Lagrangian does not depend on time explicitly, in which case H is conserved. However it is not necessarily equal to the total energy, which might not be conserved (as in the example of the bead on rotating circular hoop - see Application II in the next lecture).

2.6 Applications

Key concepts: effective potential.
Some general remarks.

Remark 2.6.1 *In Applications I - V the main strategy is to exploit the constraints and symmetries in order to reduce the problem to a 1-d problem, that is to a single parameter problem (one degree of freedom/generalised coordinate q). The intermediate step is to identify effective potential $V_{\mathit{eff}}(q)$ and derive the 2nd order differential equation for q:*

$$\ddot{q} = -\frac{\partial V_{\mathit{eff}}(q)}{\partial q}, \tag{1}$$

which has a form of Newton's equation. The form of $V_{\mathit{eff}}(q)$ in each particular problem will determine the solution. We then either solve (1) or determine main qualitative and quantitative features of the solution by analysing $V_{\mathit{eff}}(q)$.

Remark 2.6.2 *In Applications I, II we are interested in the motion of the bead relative to the hoop. We start with writing the Cartesian coordinates (x, y, z) of the bead (in the* **rest** *frame/frame of stationary observer - not rotating frame), which we parameterize by suitably chosen generalised coordinate. In both cases the constraint is* **rheonomic** *(time dependent).*

Application I: Bead on a square horizontal rotating hoop *(based on Dynamics&Relativity, Example 3, Problem 8.)*

A square hoop $ABCD$ is made of fine smooth wire and has side length $2a$. The hoop is horizontal and rotating with constant angular speed ω about a vertical axis through A. A small bead which can slide on the wire is initially at rest at the midpoint of the side BC. Let ξ be the distance of the bead from the vertex B on the side BC. Determine the second order differential equation for parameter ξ. Solve this equation. Is it possible to describe the motion of the bead qualitatively without solving the equation of motion? (Hint: what is the form of the effective potential?)

(This example will be discussed in detail during the lecture.)

Application II: Bead on circular vertical rotating hoop *(first part is addressed in Example Sheet 1, Problem 2)*

We start by parameterizing the constraint by writing the coordinates of the bead in terms of ψ and t:

$$\begin{cases} x = a \sin\psi \cos\omega t \\ y = a \sin\psi \sin\omega t \\ z = a(1 - \cos\psi). \end{cases} \tag{2}$$

This is a rheonomic (t-dependent) constraint.

In the example question you are asked to derive 2nd order DE for ψ, so I will skip this step. The resulting equation can be brought to the form

$$\ddot{\psi} = -\frac{\partial V_{\text{eff}(\psi)}}{\partial \psi}, \tag{3}$$

where

$$V_{\text{eff}}(\psi) = -\frac{1}{2}\omega^2 \sin^2\psi - \frac{g}{a}\cos\psi. \tag{4}$$

In terms of $V_{\text{eff}}(\psi)$ the Lagrangian has a very elegant and simple form

$$L(\psi, \dot{\psi}) = ma^2 \left(\frac{\dot{\psi}^2}{2} - V_{\text{eff}}(\psi) \right). \tag{5}$$

Let us analyse the behaviour of the system by examining the form of $V_{\text{eff}}(\psi)$ starting with determining stationary solutions, which correspond to $\ddot{\psi} = \dot{\psi} = 0$, i.e.

$$\frac{\partial V_{\text{eff}}}{\partial \psi} = 0 \quad \Rightarrow \quad g \sin\psi = a\omega^2 \sin\psi \cos\psi. \tag{6}$$

Thus, there are three stationary solutions:

(1) $\psi = 0$: always exists; stable for $\omega < \sqrt{g/a}$, unstable for $\omega \geq \sqrt{g/a}$.

(2) $\psi = \pi$: always exists; unstable.

(3) $\psi = \arccos(g/a\omega^2)$: exists only if the hoop is spinning fast enough, $\omega \geq \sqrt{g/a}$; stable. This is the interesting solution amongst the three.

It follows that if the hoop is spinning very fast, then $\psi = \pi/2$, which is what we expect.

An important remark: Lagrangian does not depend on time and therefore the Hamiltonian

$$H = \dot{\psi}\frac{\partial L}{\partial \dot{\psi}} - L = \frac{ma^2}{2}\left(\dot{\psi}^2 - \omega^2 \sin^2\psi\right) - mga\cos\psi$$

is conserved. The last term in the RHS is obviously the potential energy. However, the first term is *not* kinetic energy. The kinetic energy is

$$T = \frac{ma^2}{2}\left(\dot{\psi}^2 + \omega^2 \sin^2\psi\right).$$

This is a typical situation corresponding to *rheonomic* constraint, that is time dependent constraint with L which is not explicitly time dependent. The bead on rotating hoop is an excellent example of such a system. H is conserved, but total energy is not. There is a flow of energy from outside, e.g. from the motor which rotates the hoop. In this system the change in kinetic energy of the bead does not account for the change in potential energy. To maintain the angular velocity of rotation constant we need to supply energy from outside.

Application III: Spherical Pendulum
Particle of mass m is constrained to move on a surface of a sphere of constant radius l. The constraint is scleronomic. The suitable generalised coordinates are the polar, θ, and azimuthal, ϕ, angles (for convenience we will measure θ from the downward direction).

In Cartesian coordinates of the particle the Lagrangian is

$$L(z,\dot{x},\dot{y},\dot{z}) = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz. \tag{7}$$

Parametrisation of the constraint

$$\begin{cases} x = & l\cos\phi\sin\theta t \\ y = & l\sin\phi\sin\theta t \\ z = & -l\cos\theta, \end{cases} \tag{8}$$

yields

$$L(\theta,\dot{\theta},\dot{\phi}) = \frac{ml^2}{2}(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) + mgl\cos\theta. \tag{9}$$

Now, we identify the first conserved quantity: ϕ is ignorable, therefore the corresponding generalised momentum

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi}\sin^2\theta \tag{10}$$

is conserved. This is in fact, z-component of the angular momentum. Indeed, its conservation follows from rotational symmetry around z-axis.

Now we can write E-L equation for θ:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = ml^2\ddot{\theta} - ml^2\dot{\phi}^2\sin\theta\cos\theta + mgl\sin\theta, \tag{11}$$

from which we can obtain an equation for θ by substituting for $\dot{\phi}$ in terms of p_ϕ and θ from (10). The value of p_ϕ is fixed by the initial conditions of the problem. This yields the familiar form

$$\ddot{\theta} = -\frac{\partial V_{\text{eff}(\theta)}}{\partial\theta}, \tag{12}$$

where

$$V_{\text{eff}}(\psi) = \frac{p_\phi^2}{2m^2l^4\sin^2\theta} - \frac{g}{l}\cos\theta. \tag{13}$$

Now we identify the second conserved quantity. As $\partial L/\partial t = 0$, the Hamiltonian is conserved. In lecture 9 we will prove a theorem, stating that for scleronomic constraints Hamiltonian equals total energy. Thus,

$$H = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial\dot{\phi}} - L = \frac{1}{2}ml^2\dot{\theta}^2 + ml^2V_{\text{eff}} = \tilde{E}, \tag{14}$$

where the total energy $\tilde{E}$ is conserved. Hence we obtain an elegant equation for rescaled total energy:

$$\frac{\dot{\theta}^2}{2} + V_{\text{eff}}(\theta) = E. \tag{15}$$

For given E the motion is restricted to $V_{\text{eff}} \leq E$, i.e. to the range between angles θ_1 and θ_2, which are solutions of $V_{\text{eff}} = E$.

The ultimate aim is to solve for $\theta(t)$ and $\phi(t)$ or $\phi(\theta)$. One of the possible ways to proceed is to rearrange (15) for

$$dt = \frac{d\theta}{\sqrt{2(E - V_{\text{eff}})}}, \tag{16}$$

then substitute it into (10) to obtain

$$\phi(\theta) = \frac{p_\phi}{\sqrt{2ml^2}} \int \frac{d\theta}{\sqrt{(E - V_{\text{eff}})} \sin^2\theta}. \tag{17}$$

This leads to elliptic integrals of 2nd and 3rd type.

Notice that we used the constraint to reduce the number of degrees of freedom to two, then two conserved quantities to solve the problem.

Application IV: Two-body Problem *(not discussed during the lecture: handout available)*

Application V: Charged particle in a background electro-magnetic field *(This overlaps with example sheet 1)*

Consider a particle with charge e and mass m moving in the e-m field. We will postulate the form of Lagrangian for this case and show that corresponding Euler-Lagrange equation leads to Lorentz force law $m\ddot{\boldsymbol{r}} = e\boldsymbol{E} + e\,\dot{\boldsymbol{r}} \times \boldsymbol{B}$.
Working in SI units we write the fields in terms of vector, $\boldsymbol{A}(\boldsymbol{r}, t)$, and scalar, $\phi(\boldsymbol{r}, t)$, potentials as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}.$$

The Lagrangian of a charged particle in e-m field is

$$L = \frac{m\dot{\boldsymbol{r}}^2}{2} - e(\phi - \dot{\boldsymbol{r}} \cdot \boldsymbol{A}). \tag{18}$$

Notice that the form of the momentum

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + e\boldsymbol{A}.$$

E-L equation yields

$$m\ddot{\boldsymbol{r}} = -e\frac{\partial \boldsymbol{A}}{\partial t} - e\nabla\phi + e\nabla(\dot{\boldsymbol{r}} \cdot \boldsymbol{A}). \tag{19}$$

Working in components we recover Lorentz force law.

Gauge invariance: Consider Gauge transformation

$$\begin{aligned} \phi &\to \phi - \frac{\partial \Lambda}{\partial t}, \\ \boldsymbol{A} &\to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda, \end{aligned} \tag{20}$$

for any function $\Lambda(\boldsymbol{r}, t)$. How does it affect the Lagrangian?

$$L \to L + e\frac{\partial \Lambda}{\partial t} + e\dot{\boldsymbol{r}}\nabla\Lambda = L + e\frac{d\Lambda}{dt}. \tag{21}$$

But the last term is a full t-derivative of an arbitrary function of coordinates and time, hence L is gauge invariant.