Lectures 9 and 10 (28th and 30th of October 2014)

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2.7 Quadratic Lagrangians; Small Oscillations, Stability and Normal Modes

Key concepts: equilibrium, normal modes

Consider a general form of the kinetic energy term in Lagrangian, which results from the transformation from Cartesian to generalised coordinates.

Case 1 - Scleronomic constraints: $x_k = x_k(\mathbf{q})$ and therefore

$$\dot{x}_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i} \dot{q}_i.$$

Thus,

$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \dot{x}_k^2 = \frac{1}{2}\sum_{i,j}\left(\sum_k m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}\right)\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j} T_{ij}(q)\,\dot{q}_i\dot{q}_j, \tag{1}$$

where the (symmetric) matrix $T_{ij}(q)$ depends on coordinates only. Thus, T is still quadratic in velocity, but can also depend on coordinates.

An excellent example is a double pendulum (see Example sheet 1), where

$$L = \frac{1}{2}\left[(m_1+m_2)l_1^2\dot{\theta}_1^2 + m_2 l_2^2\dot{\theta}_2^2 + 2m_2 l_1 l_2 \cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] + (m_1+m_2)gl_1\cos\theta_1 + m_2 g l_2 \cos\theta_2. \tag{2}$$

Thus,

$$T_{11} = (m_1+m_2)l_1^2, \quad T_{22} = m_2 l_2^2, \quad T_{12} = T_{21} = m_2 l_1 l_2 \cos(\theta_1-\theta_2)$$

Corollary 2.7.1 *For closed and scleronomous systems Hamiltonian equals total energy of the system.*

Proof From (1) it follows that T is a homogeneous function of degree 2 in $\dot{q}$. Indeed, $T(\alpha\dot{q}) = \alpha^2 T(\dot{q})$, $\forall\ \alpha \in \mathbb{R}$.

By *Euler's Theorem for homogeneous functions*

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T.$$

Thus,

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - L = T + V = E.$$

□

Case 2 - Rheonomic constraints: $x_k = x_k(\mathbf{q}, \mathbf{t})$ and therefore

$$\sum_k \dot{x}_k^2 = \sum_{i,j}\left(\sum_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}\right)\dot{q}_i\dot{q}_j + 2\sum_i\left(\sum_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial t}\right)\dot{q}_i + \sum_k\left(\frac{\partial x_k}{\partial t}\right)^2,$$

that is T is not homogeneous in $\dot{q}$. Lagrangian might still not depend on time explicitly, in which case Hamiltonian will be conserved. However, total energy will not be conserved (as in the example of the bead on rotating hoop - application II).

Purely Kinetic Lagrangians in GR:

In General Relativity the potential is incorporated in the geometry of space-time through the metric. Hence, Lagrangians in GR do not have a potential term - they are purely kinetic Lagrangians, describing particle which moves in curved space-time. The most general form of such Lagrangian is discussed in Example Sheet 1, Problem 11.)

Small Oscillations, Stability and Normal Modes

We consider equations of motion of a physical system with n degrees of freedom. The system is said to be at *equilibrium* at $\boldsymbol{q} = \boldsymbol{q}^0$ if $\dot{\boldsymbol{q}}^0 = \boldsymbol{0}$ and $\boldsymbol{q}^0$ is the critical (stationary) point of the potential, i.e.

$$\left.\frac{\partial V(\boldsymbol{q})}{\partial q_i}\right|_{\boldsymbol{q}^0} = 0 \quad \forall\, i.$$

The point $\boldsymbol{q}^0$ is an equilibrium point. If the system is initially at equilibrium, then it will continue to be at equilibrium indefinitely. An equilibrium point is classified as stable if a small disturbance from equilibrium results only in a small bounded motion about the equilibrium.

We are now ready to write the Lagrangian in the form

$$L = \frac{1}{2} T_{ij}(q)\dot{q}_i\dot{q}_j - V(q). \tag{3}$$

We may consider small perturbations away from the equilibrium, $q_i(t) = q_i^0 + \eta_i(t)$, and linearise L in the neighbourhood of q_i^0. As a result we obtain equation of motion expanded in linear order in η.

Alternatively, we may introduce linearization once the equations of motion have been obtained. From Euler-Lagrange equation we obtain

$$T_{ij}(q)\ddot{q}_j = -\frac{\partial V(q)}{\partial q_i}. \tag{4}$$

Now, consider small deviations η_i, so $q_i = q_i^0 + \eta_i$, and

$$T_{ij}(q)\ddot{\eta}_j = -\frac{\partial V(q)}{\partial \eta_i}. \tag{5}$$

and expand T_{ij} in linear order in η, while $V(q)$ in quadratic order.

$$T_{ij}(q) = T_{ij}(q^0) + \left.\frac{\partial T_{ij}}{\partial q_k}\right|_{\boldsymbol{q}^0} \eta_k + O(\eta^2), \tag{6}$$

where it is clear why we don't need quadratic terms - they clearly will not contribute as T will be multiplied by $\ddot{\eta}$ (in fact we don't need the linear order either). For $V(q)$, however it is essential to include quadratic the term.

$$V(q) = V(q^0) + \left.\frac{\partial V(q)}{\partial q_k}\right|_{\boldsymbol{q}^0} \eta_k + \frac{1}{2} \left.\frac{\partial^2 V(q)}{\partial q_k \partial q_l}\right|_{\boldsymbol{q}^0} \eta_k \eta_l + O(\eta^3). \tag{7}$$

Here the second term on RHS vanishes as $\boldsymbol{q}^0$ is the point of minimum of V.
Thus

$$\frac{\partial V(q)}{\partial \eta_i} = \left.\frac{\partial^2 V(q)}{\partial q_i \partial q_k}\right|_{\boldsymbol{q}^0} \eta_k. \tag{8}$$

Therefore (5) becomes

$$T_{ij}(q^0)\ddot{\eta}_j = -V_{ij}\eta_j, \tag{9}$$

were

$$V_{ij} = \left.\frac{\partial^2 V(q)}{\partial q_i \partial q_j}\right|_{\boldsymbol{q}^0}. \tag{10}$$

In the matrix form

$$T\ddot{\boldsymbol{\eta}} = -V\boldsymbol{\eta}, \tag{11}$$

where both matrices are symmetric.
Thus,

$$\ddot{\boldsymbol{\eta}} = -T^{-1}V\boldsymbol{\eta}, \quad \text{and} \quad F = -T^{-1}V. \tag{12}$$

We have reduced the linearised equations of motion of a system with n degrees of freedom to the form $\ddot{\boldsymbol{\eta}} = F\boldsymbol{\eta}$. It is not very difficult to prove that F has real eigenvalues.[1] Consider an eigenvector $\boldsymbol{\mu}$ of matrix F with eigenvalue λ^2.

$$F\boldsymbol{\mu} = \lambda^2\boldsymbol{\mu} \quad \Rightarrow \quad TT^{-1}V\boldsymbol{\mu} = -\lambda^2 T\boldsymbol{\mu} \quad \Rightarrow \quad V\boldsymbol{\mu} = -\lambda^2 T\boldsymbol{\mu} \quad \Rightarrow \quad \boldsymbol{\mu}^* V\boldsymbol{\mu} = -\lambda^2\boldsymbol{\mu}^* T\boldsymbol{\mu}, \tag{13}$$

were $\boldsymbol{\mu}^* V\boldsymbol{\mu} \in \mathbb{R}$, $\boldsymbol{\mu}^* T\boldsymbol{\mu} \in \mathbb{R}$, because T and V are symmetric. Also $detT \neq 0$ since T is invertible. Thus, $\lambda^2 \in \mathbb{R}$ and this completes the proof.

Thus, consider eigenvalues and eigenvectors of F:

$$F\boldsymbol{\mu}_{(a)}(t) = \lambda_a^2\boldsymbol{\mu}_{(a)}(t), \tag{14}$$

where λ_a^2 are real and $\boldsymbol{\mu}_{(a)}$ are linearly independent (index a labels the eigenvectors, not their components). We can write *characteristic equation*

$$\ddot{\boldsymbol{\mu}}_{(a)}(t) - \lambda_a^2\boldsymbol{\mu}_{(a)}(t) = 0. \tag{15}$$

The most general solution is

$$\boldsymbol{\eta}(t) = \sum_a \boldsymbol{\mu}_{(a)}\left(A_a e^{+\lambda_a t} + B_a e^{-\lambda_a t}\right), \tag{16}$$

where A_a and B_a are integrating constants ($2n$ in total). Consider two cases.

1. Case 1: $\lambda_a^2 < 0$, i.e. it can be written as $\lambda_a = i\omega_a$, where $\omega_a \in \mathbb{R}$ (frequency). Here (15) takes the form of harmonic oscillator, $\ddot{\boldsymbol{\mu}}_{(a)}(t) + \omega_a^2\boldsymbol{\mu}_{(a)}(t)$. This corresponds to stability of the corresponding direction $\boldsymbol{\mu}_a$ in the vector space of eigenvectors.

2. Case 2: $\lambda_a^2 > 0$, i.e. $\lambda_a \in \mathbb{R}$. This corresponds to linear instability in $\boldsymbol{\mu}_{(a)}$-direction.

The eigenvectors $\boldsymbol{\mu}_{(a)}$ are called *normal modes* and *are not associated with a single degree of freedom of the system.* Equilibrium point $\boldsymbol{q}^0$ is *only* stable if $\lambda_a^2 < 0$ for all $a = 1, ..., n$. In this case, the system will typically oscillate around the equilibrium point as a linear superposition of all the normal modes, each at a different frequency. Normal modes are the new coordinates in which oscillations are decoupled. We will illustrate this with the example of the double pendulum - see Lagrangian (2).

[1] Note that F is not necessarily symmetric.