Questions and corrections are welcome: write to Berry Groisman on bg268@

The Legendre Transform

During the lectures we have introduced the Hamiltonian as a *Legendre Transform* of the Lagrangian with respect to $\dot{q}$. The purpose of this handout is to provide a general description of the method.

Consider a function $f(x, y)$ of two variables, x and y. Define a new function as follows

$$g(u, x, y) = ux - f(x, y), \tag{1}$$

where u is treated as an independent variable. Let us write the full differential of g:

$$\begin{aligned} dg &= \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy \\ &= xdu + udx - \frac{\partial f}{\partial x}dx - \frac{\partial f}{\partial y}dy. \end{aligned} \tag{2}$$

Now, let us choose the new variable as follows:

$$u = \frac{\partial f}{\partial x}, \tag{3}$$

which yields

$$\begin{aligned} dg &= xdu - \frac{\partial f}{\partial y}dy \\ &= \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy. \end{aligned} \tag{4}$$

Thus, g can be treated as a function of y and u, i.e. $g = g(y, u)$ with $\frac{\partial g}{\partial u} = x$ and $\frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y}$.

Thus, we have constructed the new function $g(y, u)$ by inverting (3) to obtain $x(u, y)$ and substituting it into the definition of g, (1). No information has been lost in the process. The new function is equivalent to the old one. In Hamiltonian formalism we take advantage of the Legendre transform method by choosing $\dot{q}$ for x, which gives $p = \frac{\partial L}{\partial \dot{q}}$ for u.