4.6 Monodromy: the reason for the logarithmic term in the solution

Recall that for

$$w'' + p(z)w' + q(z)w = 0, \tag{1}$$

in the vicinity of a RSP at $z = 0$ (WLOG - other points obtained by translation), there exist two linearly independent solutions of (1), $w_1(z)$ and $w_2(z)$, such that if $\sigma_1 = \sigma_2$, then

$$w_1(z) = z^{\sigma_1} u_1(z) \quad \text{and} \quad w_2(z) = w_1(z) \log z + z^{\sigma_1} u_2(z),$$

where u_1, u_2 are analytic for $|z| < R$ (and possibly in a larger disk), and $u_1(a) \neq 0 \neq u_2(a)$.

Let us see how the need in the lograithmic term arises:

Let $\mathcal{D} \in \{z : 0 < |z| < R\}$ be the largest open punctured disk around $z = 0$ that does not contain other singularities.

Let $z_0 \in \mathcal{D}$ (arbitrary point inside the disk). Since z_0 is an *ordinary point* of (1), there exists an open disk $\mathcal{D}_0 \subset \mathcal{D}$, in which (1) has analytic solutions which form a vector space of dimension 2. We choose LI solutions $w_1(z)$ and $w_2(z)$.

Let $\hat{w}_1(z) = w_1(e^{2\pi i} z)$ and $\hat{w}_2(z) = w_2(e^{2\pi i} z)$ be the results of an AC of w_1 and w_2 around the circle $C = \{|z| = |z_0|\}$ back to $\mathcal{D}_0$ obtained using a sequence of disks. Clearly, $\hat{w}_1(z)$ and $\hat{w}_2(z)$ are also LI solutions of (1) and hence can be written as a linear combination of $w_1(z)$ and $w_2(z)$, so there is a (constant) non-singular matrix M, such that

$$\begin{pmatrix} w_1(e^{2\pi i} z) \\ w_2(e^{2\pi i} z) \end{pmatrix} = M \begin{pmatrix} w_1(z) \\ w_2(z) \end{pmatrix},$$

where M is known as *Monodromy matrix.*

M can be brought into one of the two Jordan normal forms:

$$(i) \begin{pmatrix} e^{2\pi i\sigma} & 0 \\ 0 & e^{2\pi i\sigma'} \end{pmatrix} \quad \text{or} \quad (ii) \begin{pmatrix} e^{2\pi i\sigma} & 0 \\ 1 & e^{2\pi i\sigma} \end{pmatrix},$$

<u>In both cases</u> $w_1(e^{2\pi i} z) = e^{2\pi i\sigma} w_1(z)$. Let $g(z) = z^{-\sigma} w_1(z)$, then under the AC

$$g(e^{2\pi i} z) = (e^{2\pi i} z)^{-\sigma} w_1(e^{2\pi i} z) = z^{-\sigma} w_1(z) = g(z),$$

hence $g(z)$ is single-valued on $\mathcal{D}$ and hence can be written as a Laurent series, so

$$w_1(z) = z^\sigma \sum_{n=-\infty}^{\infty} a_n z^n.$$

By the same argument, in case (i),

$$w_2(z) = z^{\sigma'} \sum_{n=-\infty}^{\infty} b_n z^n.$$

Now the second solution in case (ii):

We have

$$w_2(e^{2\pi i} z) = w_1(z) + e^{2\pi i\sigma} w_2(z).$$

Let

$$f(z) = z^{-\sigma} w_2(z) - \frac{e^{-2\pi i\sigma} z^{-\sigma} \log z}{2\pi i} w_1(z),$$

then under the AC

$$f(e^{2\pi i} z) = f(z),$$

hence $f(z)$ is single-valued on $\mathcal{D}$ and hence can be written as a Laurent series, so (after rescaling $w_1(z)$ by the factor $2\pi i e^{2\pi i\sigma}$) we obtain

$$w_2(z) = w_1(z) \log z + z^\sigma \sum_{n=-\infty}^{\infty} b_n z^n,$$

as expected.