Solving ODE by integral representation: general method

For equations of the form[1]

$$aw'' + bw' + cw = 0, \tag{2}$$

where $w = w(z)$ and a, b, c are low-order polynomials in z (we require the latter, so we could use integration by parts), set

$$w(z) = \int_\gamma K(z,t) f(t) dt,$$

substitute into Eq. (2) and integrate by parts to eliminate the terms with z^n. The factor $K(z,t)$ is called a *kernel.* Three following kernels are commonly used:

$$\begin{array}{rl} e^{zt} & \text{(Laplace kernel)} \\ (z-t)^\gamma & \text{(Euler kernel)} \\ t^z & \text{(Mellin kernel)} \end{array}$$

An example of Euler kernel:

$$w(z) = \int_\gamma t^{a-c}(1-t)^{c-b-1}(t-z)^{-a} dt,$$

which satisfies the *hypergeometric equation*

$$z(1-z)w'' + [c - (a+b-1)z]\, w' - ab\, w = 0,$$

with constant a, b, c, provided that

$$\left[t^{a-c+1}(1-t)^{c-b}(t-z)^{1-a}\right]_\gamma = 0.$$

The integrand might have branch points, possibly at $t = 0, 1, z$, depending on the values of a, b, c and on whether the exponents are integers. If there are any branch cuts, then γ must not cross them, of course.[2]

[1] See, for example Paper 1, q.11 in 2015 Tripos exam for a specific example.
[2] See related question 6 on Example Sheet 3