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Parametrization of the rotation matrix R

The purpose of this handout is to clarify the issue of the parametrization of the rotation matrix R (as defined in Lectures 10 and 14) in terms of the three Euler angles, ϕ, θ and ψ. [1]
We have defined matrix R as the matrix which rotates the space frame $\{\tilde{\boldsymbol{e}}_a\}$ to the body frame $\{\boldsymbol{e}_a\}$, i.e.

$$\boldsymbol{e}_a = R_{ba}\tilde{\boldsymbol{e}}_b \tag{1}$$

The same matrix links the coordinates of position vectors in body frame with their coordinates in the space frame as follows

$$\tilde{\boldsymbol{r}} = R\boldsymbol{r}. \tag{2}$$

We must be conscious of the above difference: in the former case we are rotating the basis, while in the latter we are rotation coordinates. For inverse transformations from space coordinates to body coordinates and from body frame to space frame we use transposed matrix,

$$\boldsymbol{r} = R^{-1}\tilde{\boldsymbol{r}}, \qquad\qquad \tilde{\mathbf{e}}_a = R_{ab}\mathbf{e}_b. \tag{3}$$

Thus, the rotation matrix can be defined in two different ways and it is worth keeping in mind when reading textbooks.

Let us derive R by explicit construction. We are starting with the body frame, which is initially aligned with the space frame and perform three consecutive rotations. (Below we use the following notation: $R(\hat{\mathbf{a}}, \alpha)$ is a rotation by angle α about unit axis $\hat{\mathbf{a}}$.)

(1) $R(\tilde{\boldsymbol{e}}_3, \phi)$, i.e. rotation by angle ϕ about the third axis of the space frame, $\tilde{\boldsymbol{e}}_3$. This results in the new frame $\{\boldsymbol{e}'_a\}$. (Notice that $\mathbf{e}'_3 = \tilde{\mathbf{e}}_3$.)

(2) $R(\boldsymbol{e}'_1, \theta)$, i.e. rotation by angle θ about the axis $\boldsymbol{e}'_1$ of the new frame. We obtain a new frame, $\{\boldsymbol{e}''_a\}$ as a result. (Notice that $\mathbf{e}''_1 = \mathbf{e}'_1$.)

(3) $R(\boldsymbol{e}''_3, \psi)$, i.e. rotation by angle ψ about the axis $\boldsymbol{e}''_3$ of the new frame. (In fact $\boldsymbol{e}''_3$ is already the axis $\boldsymbol{e}_3$ of the body frame.) This accomplishes the entire rotation.

The resulting matrix is $R = R(\boldsymbol{e}''_3, \psi)R(\boldsymbol{e}'_1, \theta)R(\tilde{\mathbf{e}}_3, \phi)$.

In order to determine the form of this matrices in space frame let us make the following observation [2]. For two axes $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$

$$R\left(R(\hat{\boldsymbol{a}}, \alpha)\hat{\boldsymbol{b}}, \beta\right) R(\hat{\boldsymbol{a}}, \alpha) = R(\hat{\boldsymbol{a}}, \alpha)R(\hat{\boldsymbol{b}}, \beta). \tag{4}$$

I will omit the proof of the above statement, but you can convince yourself by simple geometric considerations. In fact, (4) [with the second factor on the LHS taken over to the RHS] is just the statement that the

[1] I am grateful to Clive Wells for helping me to clarify this issue. This handout is at large based on my discussions with Clive and material provided by him.

[2] There are several ways to go about constructing R. One alternative will be to consider 3-step back-substitution of the bases.

rotation matrix is a tensor, determined by the vector (which transforms as vector).

In particular, consider the two consecutive rotations $R(\boldsymbol{e}_1', \theta)R(\tilde{\boldsymbol{e}}_3, \phi)$. Observe, that this is equivalent to rotation by θ about $\tilde{\boldsymbol{e}}_1$ followed by rotation by ϕ about $\tilde{\boldsymbol{e}}_3$. In other words,

$$\boldsymbol{e}_1' = R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\boldsymbol{e}}_1, \tag{5}$$

and therefore

$$R(\boldsymbol{e}_1', \theta)R(\tilde{\boldsymbol{e}}_3, \phi) = R\left(R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\boldsymbol{e}}_1, \theta\right) R(\tilde{\boldsymbol{e}}_3, \phi) = R(\tilde{\boldsymbol{e}}_3, \phi)R(\tilde{\boldsymbol{e}}_1, \theta). \tag{6}$$

Similarly,

$$\begin{aligned} \boldsymbol{e}_3'' = R(\boldsymbol{e}_1', \theta)\mathbf{e}_3' &= R(\boldsymbol{e}_1', \theta)R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\mathbf{e}}_3 \\ &= R(R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\boldsymbol{e}}_1, \theta)R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\mathbf{e}}_3, \end{aligned} \tag{7}$$

hence

$$\begin{aligned} R(\boldsymbol{e}_3'', \psi) &= R\left(R(R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\boldsymbol{e}}_1, \theta)R(\tilde{\boldsymbol{e}}_3, \phi)\tilde{\mathbf{e}}_3, \psi\right) \\ &= R\left(R(\tilde{\boldsymbol{e}}_3, \phi)R(\tilde{\boldsymbol{e}}_1, \theta)\tilde{\mathbf{e}}_3, \psi\right) \end{aligned} \tag{8}$$

Putting (6) and (8) together

$$\begin{aligned} R &= R\left(R(\tilde{\boldsymbol{e}}_3, \phi)R(\tilde{\boldsymbol{e}}_1, \theta)\tilde{\mathbf{e}}_3, \psi\right) R(\tilde{\boldsymbol{e}}_3, \phi)R(\tilde{\boldsymbol{e}}_1, \theta) \\ &= R(\tilde{\mathbf{e}}_3, \phi)R(\tilde{\mathbf{e}}_1, \theta)R(\tilde{\boldsymbol{e}}_3, \psi). \end{aligned} \tag{9}$$

The three latter rotations are characterised by the following rotation matrices.

$$R(\tilde{\boldsymbol{e}}_3, \phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R(\tilde{\boldsymbol{e}}_1, \theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R(\tilde{\boldsymbol{e}}_3, \psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the resulting matrix is

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\sin\psi\cos\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix} \tag{10}$$