

Small Oscillations, Stability and Normal Modes:

Proof that matrix F has real eigenvalues

In lecture we reduced the linearised equations of motion of a system with n degrees of freedom to the form $\ddot{\eta} = F\eta$ and claimed that matrix F has real eigenvalues. Here we are going to prove this statement. Let us start with Lagrangian

$$L = \frac{1}{2}T_{ij}(q)\dot{q}_i\dot{q}_j - V(q), \quad (1)$$

and from Euler-Lagrange equation obtain

$$T_{ij}(q)\ddot{q}_j = -\frac{\partial V(q)}{\partial q_i}. \quad (2)$$

Now, consider small deviations η_i , so $q_i = q_i^0 + \eta_i$, and

$$T_{ij}(q)\ddot{\eta}_j = -\frac{\partial V(q)}{\partial \eta_i}. \quad (3)$$

and expand T_{ij} in linear order in η , while $V(q)$ in quadratic order.

$$T_{ij}(q) = T_{ij}(q^0) + \left. \frac{\partial T_{ij}}{\partial q_k} \right|_{\mathbf{q}^0} \eta_k + O(\eta^2), \quad (4)$$

where it is clear why we don't need quadratic terms - they clearly will not contribute as T will be multiplied by $\ddot{\eta}$ (in fact we don't need the linear order either). For $V(q)$, however it is essential to include quadratic term.

$$V(q) = V(q^0) + \left. \frac{\partial V(q)}{\partial q_k} \right|_{\mathbf{q}^0} \eta_k + \frac{1}{2} \left. \frac{\partial^2 V(q)}{\partial q_k \partial q_l} \right|_{\mathbf{q}^0} \eta_k \eta_l + O(\eta^3). \quad (5)$$

Here the second term on RHS vanishes as \mathbf{q}^0 is the point of minimum of V .

Thus

$$\frac{\partial V(q)}{\partial \eta_i} = \left. \frac{\partial^2 V(q)}{\partial q_i \partial q_k} \right|_{\mathbf{q}^0} \eta_k. \quad (6)$$

Therefore (3) becomes

$$T_{ij}(q^0)\ddot{\eta}_j = -V_{ij}\eta_j, \quad (7)$$

where

$$V_{ij} = \left. \frac{\partial^2 V(q)}{\partial q_i \partial q_j} \right|_{\mathbf{q}^0}. \quad (8)$$

In the matrix form

$$T\ddot{\eta} = -V\eta. \quad (9)$$

Thus,

$$\ddot{\eta} = -T^{-1}V\eta, \quad \text{and} \quad F = -T^{-1}V. \quad (10)$$

Now everything is ready for the final step. Consider an eigenvector μ of F with eigenvalue λ^2 :

$$F\mu = \lambda^2\mu \Rightarrow TT^{-1}V\mu = -\lambda^2T\mu \Rightarrow V\mu = -\lambda^2T\mu \Rightarrow \mu^*V\mu = -\lambda^2\mu^*T\mu, \quad (11)$$

where $\mu^*V\mu \in R$, $\mu^*T\mu \in R$, because T and V are symmetric. Also $\det T \neq 0$ since T is invertible. Thus, $\lambda^2 \in R$ and this completes the proof.