4.3 Solutions of 2nd-order ODE near ordinary and regular singular points of

$$w'' + p(z)w' + q(z)w = 0, \tag{1}$$

where $p(z)$, $q(z)$ and $w(z)$ are meromorphic on $\mathbb{C}$.

Solutions near an ordinary point (OP)

Theorem: If $p(z)$ and $q(z)$ are analytic in the disk $|z| < R$ ($z = 0$ is an OP), then there exist two linearly independent solutions to the Eqn. (1), $w_1(z)$ and $w_2(z)$, such that

(i) w_1, w_2 are analytic in $|z| < R$ (and possibly in a larger disk);

(ii) $w_1(0) \neq 0$, $w_2(0) = 0$, but $w_2'(0) \neq 0$ (i.e. the roots of the indicial equation are 0 and 1).

Proof (outline - non-examinable): By substituting $p(z) = \sum_{n=0}^{\infty} p_n z^n$, $q(z) = \sum_{n=0}^{\infty} q_n z^n$ and $w(z) = \sum_{n=0}^{\infty} a_n z^n$ into (1), equating coefficients of z^n and checking the radius of convergence. (Do not attempt it. There are better ways, e.g. google Picard iteration.)

Remarks:

- Note that (for example) z and z^2 cannot both satisfy an equation of the form (1), for which $z = 0$ is an OP.
- An example of an equation with solutions that are analytic in a disk larger than the disk in which $p(z)$ and $q(z)$ are analytic:

  $$w'' - \frac{2}{z-1}w' + \frac{2}{(z-1)^2}w = 0,$$

  which has solution $w(z) = A(z-1) + B(z-1)^2$ - an entire function despite the singular point of the equation at $z = 1$.

Solutions near a regular singular point

Suppose $zp(z)$ and $z^2 q(z)$ are analytic in the disk $|z| < R$. Let

$$p(z) = \sum_{n=-1}^{\infty} p_n z^n, \quad \text{and} \quad q(z) = \sum_{n=-2}^{\infty} q_n z^n.$$

The indicial equation is

$$\sigma^2 + (p_{-1} - 1)\sigma + q_{-2} = 0. \tag{2}$$

Eqn. (2) can be thought in connection with the $|z| \ll 1$ approximation of Eqn. (1):

$$w'' + p_{-1}z^{-1}w' + q_{-2}z^{-2}w \sim 0,$$

which has solutions

$$\begin{array}{ll} z^{\sigma_1}, z^{\sigma_2} & \text{if } \; \sigma_1 \neq \sigma_2, \\ z^{\sigma_1}, z^{\sigma_1}\log z & \text{if } \; \sigma_1 = \sigma_2, \end{array}$$

where σ_1, σ_2 are the roots of (2).

Theorem: Let $z = 0$ be a RSP of Eqn. (1). Then there exist two linearly independent solutions of (1), $w_1(z)$ and $w_2(z)$, such that

(1) If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then

$$w_1(z) = z^{\sigma_1}u_1(z) \;\text{ and }\; w_2(z) = z^{\sigma_2}u_2(z),$$

where u_1, u_2 are analytic for $|z| < R$ (and maybe in a larger disk least), and $u_1(0) \neq 0 \neq u_2(0)$.

(2) If $\sigma_1 = \sigma_2$, then

$$w_1(z) = z^{\sigma_1}u_1(z) \;\text{ and }\; w_2(z) = w_1(z)\log z + z^{\sigma_1}u_2(z),$$

where u_1, u_2 are analytic for $|z| < R$ (and maybe in a larger disk least), and $u_1(0) \neq 0 \neq u_2(0)$.

(3) If $\sigma_1 \neq \sigma_2$, but $\sigma_1 - \sigma_2 \in \mathbb{Z}$ (say $\sigma_1 > \sigma_2$), then

$$w_1(z) = z^{\sigma_1}u_1(z) \;\text{ and }\; w_2(z) = Cw_1(z)\log z + z^{\sigma_2}u_2(z),$$

where C is some constant, which may sometimes be zero.

Proof: (Case 1) Substitute $w(z) = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n$ into (1) and (assuming $a_0 \neq 0$) equate the coefficients of z^{n+z} to zero to obtain

$$\begin{cases} a_0F(\sigma) = 0, & \text{for } \; n = 0, \\ a_nF(n+\sigma) = \; -\sum_{k=0}^{n-1} a_k\left[(k+\sigma)p_{n-k-1} + q_{n-k-2}\right], & \text{for } \; n > 0, \end{cases} \tag{3}$$

where $F(x) = x(x-1) + p_{-1}x + q_{-2} = (x-\sigma_1)(x-\sigma_2)$.
Since $a_0 \neq 0$, the top equation (3) is the indicial equation. Its exponents (i.e. roots) satisfy

$$\sigma_1 + \sigma_2 = 1 - p_{-1} \quad \text{and} \quad \sigma_1\sigma_2 = q_{-2}\,.$$

If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then the second equation in (3) gives the recurrence relations for a_k, which determine two LI solutions which can be written in the form given in Case 1.

Case 2 and Case 3: e-handout to follow.