Meromorphic continuation

$\exists$ singularities so "serious" that they prevent AC of the function in question (non-isolated singularity).

Natural barrier

A prototypical example $f(z) = \sum_{n=0}^{\infty}{}' z^{2^n}$, converges $|z|<1$ (ratio test) (1)

Let's show that AC into $|z|>1$ is impossible.

- $f(z^2) = \sum_{n=0}^{\infty}{}' (z^2)^{2^n} = \sum_{n=0}^{\infty} z^{2^{n+1}} = \sum_{n=1}^{\infty}{}' z^{2^n} = f(z) - z.$ (2)

From (1), $z_0 = 1$ is a singular point: $f(1) = \infty$.

From (2), $z_1^2 = 1$ ($z_1 = \pm 1$) is also singular: $f(z_1^2) = \infty \Rightarrow f(z_1) = \infty + z_1 = \infty$.

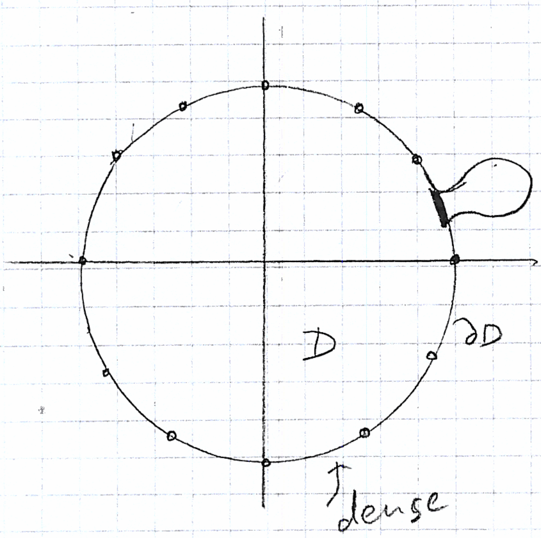
Similarly,

- $f(z^4) = \sum{}' (z^4)^{2^n} = \sum z^{2^{n+2}} = \sum_{n=2}^{\infty}{}' z^{2^n} = f(z^2) - z^2 = f(z) - z - z^2$

For $z_2^4 = 1$, ($z_2 = \pm 1, \pm i$) $f(z_2) = \infty + z_2 + z_2^2 = \infty$.

By induction; $f(z^{2^m}) = f(z) - \sum_{j=0}^{m-1}{}' z^{2^j}$

Thus, for all points z_m on the unit circle, satisfying $z^{2^m} = 1$ (all roots of unity) $f(z_m) = \infty$. All these points are singular points.



To continue $f(z)$ to $|z|>1$ at the very least we need $f(z)$ to be analytic on some small arc of the u.c. $|z|=1$.

But this is impossible, as the ROU are dense.

The boundary, ∂D, is the NATURAL BARRIER for the function f.

Applications: arise as solns of certain non-linear DEs.