THE BEST CONSTANT AND EXTREMALS OF THE
SOBOLEV EMBEDDINGS IN DOMAINS WITH HOLES:
THE $L^\infty$ CASE.

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded, convex domain. We study the
best constant of the Sobolev trace embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$ for
functions that vanish in a subset $A \subset \Omega$, which we call the hole. That is,
we deal with the minimization problem $S_A^T = \inf \|u\|_{W^{1,\infty}(\Omega)}/\|u\|_{L^\infty(\partial\Omega)}$
for functions that verify $u_{|A}=0$. We find that there exists an optimal
hole that minimizes the best constant $S_A^T$ among subsets of $\Omega$ of
prescribed volume and we give a geometrical characterization of this
optimal hole. In fact, minimizers associated to these holes are cones
centered at some points $x_0^*$ on $\partial\Omega$ and the best holes are of the form
$A^* = \Omega \setminus B(x_0^*, r^*)$.

A similar analysis can be performed for the best constant of the
embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ with holes. In this case we also find
that minimizers associated to optimal holes are cones centered at some
points $x_0^*$ on $\partial\Omega$ and the best holes are of the form $A^* = \Omega \setminus B(x_0^*, r^*)$.

1. Introduction

Sobolev inequalities are relevant for the study of boundary value problems
for differential operators. They have been studied by many authors and it is
by now a classical subject. It at least goes back to [2], for more references see
[6]. In particular, the Sobolev trace inequality has been intensively studied
in [4, 7, 8, 9, 12, 18, 20, 21], etc.

Let $\Omega$ be a bounded, convex domain in $\mathbb{R}^N$. In this paper we want to study
the best constant and extremals for the embeddings $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$
and $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ restricted among functions that vanish in a subset
$A$ of $\Omega$.

First, we deal with the trace embedding. To this end, for any function
$u \in W^{1,\infty}(\Omega)$, we define the associated quotient

$$Q^T(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}.$$
For $A \subset \Omega$ we let

$$S_{A}^{T} := \inf \left\{ Q^{T}(u), u \in W^{1,\infty}(\Omega) \text{ s.t. } u \neq 0 \text{ on } \partial \Omega, u = 0 \text{ in } A \right\}.$$ 

This constant $S_{A}^{T}$ is the best Sobolev trace constant for the embedding $W^{1,\infty}(\Omega) \hookrightarrow L^{\infty}(\partial \Omega)$ restricted to functions that vanish on a subset $A$ of $\Omega$. Since we are dealing with continuous functions we can assume that the set $A$ is closed (otherwise just consider the closure of $A$).

Variational problems in $L^{\infty}$ have been considered recently, due to several mathematical difficulties that are involved and where new phenomena have been observed, see for example, [1, 3, 17] and references therein. In particular, $L^{\infty}$ problems have been obtained as limits as $p \to \infty$ of $L^{p}$ problems, see [10, 15, 17, 19]. In those papers a PDE approach is used and the notion of viscosity solutions play a key role in most of them. However, in this paper we will not use any PDE nor take the limit as $p \to \infty$, but we use a more direct and geometric approach, taking advantage of the fact that the $L^{\infty}$ norm gives pointwise information.

Optimization problems for minima of Rayleigh quotients have been extensively studied in the literature due to the many applications in several branches of applied mathematics and engineering, specially in optimal design problems, see the survey [16]. Optimal design problems are usually formulated as problems of minimization of the energy stored in the design under a prescribed loading. For applications to engineering of optimization for Steklov eigenvalues, see [5].

In view of the above discussion, we consider the following optimization problem:

For a fixed $0 < \alpha < |\Omega|$, find a set $A^{*}$ of measure $\alpha$ that minimizes $S_{A}^{T}$ among all measurable subsets $A \subset \Omega$ of measure $\alpha$. That is,

$$S^{T}(\alpha) := \inf_{A \subset \Omega, |A| = \alpha} S_{A}^{T} = S_{A^{*}}^{T}.$$ 

In this paper we prove that there exist optimal holes $A^{*}$ (with their corresponding extremals $u^{*}$) for this optimization problem.

This optimization problem in $W^{1,p}(\Omega)$ has been considered recently. In fact, in [13] the existence of an optimal hole for the trace embedding has been established, see also [11] for numerical computations. Then, in [14], the interior regularity of optimal holes was analyzed.

Once existence of an optimal hole is proved, a natural question is what can be said about the extremals $u^{*}$ and the optimal holes $A^{*} = \{u^{*} = 0\}$.

Here we prove that minimizers associated to optimal holes are cones centered at some point $x_{0}^{*}$ on $\partial \Omega$ and the best holes are of the form $A^{*} = \Omega \setminus B(x_{0}^{*}, r^{*})$. Moreover, we find a geometrical characterization of an optimal hole (and its corresponding extremal).
To give the geometrical characterization of optimal holes, note that for any $x_0 \in \partial \Omega$ there exists a unique radius $r = r(x_0)$ defined by $|\Omega \setminus B(x_0, r)| = \alpha$.

Our main result for the trace embedding reads as follows:

**Theorem 1.1.** There exists an optimal hole $A^*$ in the sense that it minimizes $S_A^0$ among subsets of $\Omega$ with measure $\alpha$.

Moreover, every optimal hole is of the form $A^* = \Omega \setminus B(x_0^*, r^*)$, with $x_0^*$ such that

$$r^* = r(x_0^*) = \max_{x_0 \in \partial \Omega} r(x_0),$$

and the corresponding extremal is the cone

$$u^*(x) = \left(1 - \frac{|x - x_0^*|}{r^*}\right)^+.$$  

Note that for any $u \in W^{1,\infty}(\Omega)$ it holds that

$$S(|\{u = 0\}| \|u\|_{L^\infty(\partial \Omega)}) \leq \|u\|_{W^{1,\infty}(\Omega)}$$

Remark that this inequality is sharp. The function $S(\alpha)$ can be computed using our result. In fact,

$$S(\alpha) = \frac{1}{r^* + 1}, \quad r^* = r^*(\alpha, \Omega).$$

In some cases this $r^*$ can be computed explicitly. For example, let $\Omega$ be the unit cube in $\mathbb{R}^2$, $\Omega = [0, 1]^2$. It is clear that the vertex of an optimal cone must be located at one corner of the square, then we easily obtain,

$$r^* = 2\sqrt{\frac{1 - \alpha}{\pi}}, \quad \text{if } \alpha \geq 1 - \frac{\pi}{4},$$

while $r^*$ is given implicitly by

$$\sqrt{(r^*)^2 - 1} + \int_0^1 \sqrt{(r^*)^2 - x^2} \, dx = 1 - \alpha, \quad \text{if } \alpha < 1 - \frac{\pi}{4}.$$

In this case it is also clear that there exists exactly four optimal holes for each $\alpha$.

Now, we can perform a similar analysis for the usual Sobolev embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ with holes. Let

$$Q(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}.$$

For $A \subset \Omega$ we let

$$S_A := \inf \left\{ Q(u), u \in W^{1,\infty}(\Omega) \text{ s.t. } u \not\equiv 0 \text{ in } \Omega, u = 0 \text{ in } A \right\}.$$  

This constant $S_A$ is the best constant for $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ restricted to functions that vanish on a subset $A$ of $\Omega$.  


Theorem 1.2. There exists an optimal hole $A^*$ in the sense that it minimizes $S_A$ among subsets with measure $\alpha$.

Moreover, the same conclusion as in Theorem 1.1 holds. The best holes are complements of balls centered at $x_0^*$ on the boundary and the best functions are cones.

Organization of the paper: In Section 2 we deal with the Sobolev trace embedding and in Section 3 we briefly explain the main arguments for the Sobolev embedding.

2. The best Sobolev trace constant

As we have mentioned in the introduction, for $u \in W^{1,\infty}(\Omega)$ we define

$$Q^T(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}$$

and for $A \subset \Omega$,

$$S^T_A := \inf \{ Q^T(u), u \in W^{1,\infty}(\Omega) \text{ s.t. } u \not\equiv 0 \text{ on } \partial\Omega, u = 0 \text{ in } A \}.$$ 

Our first lemma shows that $S^T_A$ is attained.

Lemma 2.1. Consider $A \subset \Omega$, $|A| = \alpha$ a fixed hole in $\Omega$, then there exists $u \in W^{1,\infty}(\Omega)$ that minimizes $S^T_A$.

Proof. Consider a minimizing sequence $u_n \in W^{1,\infty}(\Omega)$. We can assume that $\|u_n\|_{L^\infty(\partial\Omega)} = 1$, if not, just consider the normalized sequence $v_n = \frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}$.

Then our sequence $u_n$ is bounded in $W^{1,\infty}(\Omega)$, as $\|u_n\|_{W^{1,\infty}(\Omega)} \leq S^T_A + 1$ for $n$ large. Therefore, using that the embedding $W^{1,\infty}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, we can extract a subsequence (that we still call $u_n$) such that

$$u_n \rightharpoonup^* u$$

weakly-$^*$ in $W^{1,\infty}(\Omega)$ and uniformly in $\overline{\Omega}$.

By the weak-$^*$ convergence we have

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \liminf \|\nabla u_n\|_{L^\infty(\Omega)},$$

and by the uniform convergence up to the boundary

$$\|u_n\|_{L^\infty(\Omega)} \to \|u\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \|u\|_{L^\infty(\partial\Omega)}.$$ 

Therefore $\|u\|_{L^\infty(\partial\Omega)} = 1$, $u = 0$ in $A$ and

$$Q^T(u) \leq \liminf Q^T(u_n).$$

It follows that $u$ is a minimizer of $S^T_A$. \qed
Next we want to show the existence of an optimal hole $A^*$ for $S_A^T$. For this we define
\begin{equation}
S^T(\alpha) = \inf_{A \subset \Omega, |A| = \alpha} S_A^T.
\end{equation}
Note that $S^T(\alpha)$ also have the following variational characterization
\begin{equation*}
S^T(\alpha) = \inf \{ Q^T(u) \text{ s.t. } u \in W^{1,\infty}(\Omega), |\{u = 0\}| \geq \alpha, u \not\equiv 0 \text{ on } \partial \Omega \}.
\end{equation*}

The next result shows that there exists an optimal hole.

**Theorem 2.1.** There exists an optimal hole $A^*$ with $|A^*| \geq \alpha$ such that $S_A^* = S^T(\alpha)$.

**Proof.** Our problem is to find extremals for (2.1).

If we consider sets $A$ with $|A| \geq \alpha$ we only extend our number of test functions and therefore
\begin{equation*}
S^T(\alpha) \geq \inf \{ S_A^T \text{ with } A \subset \Omega, |A| \geq \alpha \}.
\end{equation*}
On the other hand, if $v$ is a test function for a set of measure greater than or equal than $\alpha$ it is also a test function for $S^T(\alpha)$. Thus the two infima coincide
\begin{equation*}
S^T(\alpha) = \inf \{ S_A^T \text{ with } A \subset \Omega, |A| = \alpha \}
= \inf \{ S_A^T \text{ with } A \subset \Omega, |A| \geq \alpha \}.
\end{equation*}
Further note that we can always restrict ourselves to nonnegative test functions by changing $u$ by its absolute value.

So let $A_n$ be a minimizing sequence for $S^T(\alpha)$ with extremals $u_n$ normalized with $\|u_n\|_{L^\infty(\partial \Omega)} = 1$. Like in the proof of the previous lemma we can assume that $u_n$ converges weakly-* in $W^{1,\infty}(\Omega)$ and uniformly in $\Omega$ to a function $u \in W^{1,\infty}(\Omega)$ with $\|u\|_{L^\infty(\partial \Omega)} = 1$.

Now we have to consider the limiting set of the sequence of holes $A_n$. Since the characteristic functions of $A_n$ are bounded in $L^\infty(\Omega)$ we can extract a subsequence such that $\chi_{A_n} \rightharpoonup^* \phi$ with $0 \leq \phi \leq 1$. So that in particular, for $A = \{ \phi > 0 \}$ we have
\begin{equation*}
|A| \geq \int_\Omega \phi = \lim \int_\Omega \chi_{A_n} = \lim |A_n| \geq \alpha.
\end{equation*}
Since $u \geq 0$, $\phi \geq 0$ and
\begin{equation*}
\int_\Omega u \phi = \lim \int_\Omega u_n \chi_{A_n} = 0
\end{equation*}
we get that $u$ vanishes in $A$, where $A$ has measure $|A| \geq \alpha$. Hence, $u$ vanishes on $A^* = \overline{A}$ with $|A^*| \geq |A| \geq \alpha$. Since $u \not\equiv 0$, $\{ u > 0 \}$ is a nonempty open set and therefore $A^*$ is a proper subset of $\Omega$. 

As before, the convergence of \( u_n \) to \( u \) (in different topologies) implies that
\[
\lim \inf Q^T(u_n) \geq Q^T(u).
\]

As \( u \) is an admissible function we conclude that \( A^* \) is an optimal set and that \( u \) is an extremal for \( S^T(\alpha) \). \( \square \)

Now we want to specify properties of extremals of \( S^T_A \). We begin with the proof of the following lemma.

**Lemma 2.2.** Let \( A \subset \Omega \), \( |A| = \alpha < |\Omega| \) and \( u \) an extremal of \( S^T_A \). Then \( u \) attains its maximum on the boundary of \( \Omega \).

**Proof.** Let \( u \) be an optimal function of \( S^T_A \) for a hole \( A \subset \Omega \) with \( |A| = \alpha < |\Omega| \). Because \( u \in W^{1,\infty}(\Omega) \), \( u \) is Lipschitz continuous and therefore attains a maximum in \( \Omega \). Let \( x_0 \in \Omega \) be a point where the maximum is attained
\[
u(x_0) = \max_{x \in \Omega} u(x).
\]
As before we assume that \( u \) is normalized with \( u(x_0) = 1 \).

We want to prove that the maximum is attained at the boundary. Assume not, that is \( x_0 \in \Omega \) and \( \|u\|_{L^\infty(\Omega)} = 1 \), \( \|u\|_{L^\infty(\partial\Omega)} = k < 1 \).

Define a new function
\[
\bar{u}(x) = \begin{cases} u(x) & \text{if } u(x) \leq k, \\ k & \text{if } u(x) > k. \end{cases}
\]
So \( \bar{u} \) still vanishes on \( A \), \( \bar{u}(x) = u(x) \) for \( x \in \partial\Omega \), \( \|\nabla \bar{u}\|_{L^\infty(\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)} \) and \( \|\bar{u}\|_{L^\infty(\Omega)} \leq k < 1 = \|u\|_{L^\infty(\Omega)} \). But then it follows that
\[
Q^T(\bar{u}) = \frac{\|\nabla \bar{u}\|_{L^\infty(\Omega)} + \|\bar{u}\|_{L^\infty(\Omega)}}{\|\bar{u}\|_{L^\infty(\partial\Omega)}} < \frac{\|\nabla u\|_{L^\infty(\Omega)} + 1}{\|u\|_{L^\infty(\partial\Omega)}} = Q^T(u),
\]
which is a contradiction to our assumption that \( u \) is an extremal of \( Q^T(v) \). It follows \( u \) attains its maximum on the boundary of \( \Omega \). \( \square \)

As the problem is posed in \( W^{1,\infty}(\Omega) \) test functions are Lipschitz continuous in \( \Omega \). Therefore cones are natural candidates to evaluate the quotient \( Q^T(u) \) and then to estimate the infimum \( S^T_A \). Moreover, in the next theorem, we find that cones are extremals for \( S^T_A \).

With the knowledge that an extremal of \( S^T_A \) attains its maximum value in a point \( x_0 \) on the boundary of \( \Omega \) we can further prove that the cone with center in \( x_0 \) and radius
\[
\text{dist}(x_0, A) = \min_{y \in A} |x_0 - y|
\]
is an extremal for \( S^T_A \).
We will denote by $C_{y,t}(x)$ the cone with vertex at $y$ and slope $1/t$,
$$C_{y,t}(x) = \left(1 - \frac{|x-y|}{t}\right)^+.$$  

**Theorem 2.2.** Let $A \subset \Omega$, $|A| = \alpha < |\Omega|$ and $u$ be an extremal for $S_A^T$. Then the cone $C_{x_0,r}$ with $x_0 \in \partial \Omega$ where $u(x_0) = \max_{x \in \Omega} u(x)$ and $r = \text{dist}(x_0, A)$ is an extremal for $S_A^T$.

**Proof.** Let $u \in W^{1,\infty}(\Omega)$ be an extremal of $S_A^T$. From Lemma 2.2 we know that $u$ attains its maximum in a point $x_0 \in \partial \Omega$. Without loss of generality we assume $u(x_0) = 1$. Then it follows that
$$Q^T(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial \Omega)}}{\|u\|_{L^\infty(\partial \Omega)}} = \|\nabla u\|_{L^\infty(\Omega)} + 1.\quad(2.2)$$

Now consider an arbitrary point $y \in \partial A$. By the mean value theorem we get that
$$\frac{|u(x_0) - u(y)|}{|x_0 - y|} = |\nabla u(\xi)| \leq \|\nabla u\|_{L^\infty(\Omega)},$$
for a point $\xi$ between $x_0$ and $y$. As $u(x_0) = 1$ and $u(y) = 0$ we get
$$\frac{1}{|x_0 - y|} \leq \|\nabla u\|_{L^\infty(\Omega)}, \quad \forall y \in \partial A,$$
and hence
$$\frac{1}{r} = \frac{1}{\text{dist}(x_0, A)} \leq \|\nabla u\|_{L^\infty(\Omega)}.\quad(2.3)$$

It follows that
$$\frac{1}{r} + 1 \leq Q^T(u).\quad(2.2)$$

On the other hand, choose as a test function $v = C_{x_0,r}$. Note that $v(x_0) = \max v(x) = 1$. We obtain
$$Q^T(u) \leq Q^T(v) = \frac{\|\nabla v\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\partial \Omega)}}.$$  

Since $v$ is a cone it follows that
$$\|\nabla v\|_{L^\infty(\Omega)} = \frac{1}{r}.$$  

Therefore,
$$Q^T(u) \leq \frac{\frac{1}{r} + 1}{1} = \frac{1}{r} + 1.\quad(2.3)$$

Combining (2.2) and (2.3) we get that
$$Q^T(u) = \frac{1}{r} + 1 = Q^T(C_{x_0,r}).$$

It follows that the cone $C_{x_0,r}$ is an extremal for $S_A^T$. $\square$
Further we want to prove that the cone defined in Theorem 1.1 is an extremal for $S_T(\alpha)$ and gives an optimal hole $A^*$ as the complement of a ball in $\Omega$. As we have mentioned in the introduction for any $x_0 \in \partial \Omega$ there exists a unique radius $r = r(x_0)$ defined by $|\Omega \setminus B(x_0, r)| = \alpha$. Observe that $r$ is a continuous function on $\partial \Omega$.

Now we can proceed with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** It remains to show that every optimal hole is as described in Theorem 1.1.

Let $A^*$ be an optimal hole with measure $|A^*| \geq \alpha$. Then there exists an extremal for $S_{A^*}^T$, that is the cone $C_{x_0, r}(x)$ with $x_0 \in \partial \Omega$ and $r = \text{dist}(x_0, A^*)$.

Observe that for $C_{y, t}$ we have $Q_T(C_{y, t}) = 1/t + 1$.

So, among cones, $Q_T(C_{y, t})$ is minimized when the radius $t$ is the largest possible, that is when $y = x_0^*$ and $t = r^*$. Therefore, as $A^*$ is an optimal hole it must be of the form $A^* = \Omega \setminus B(x_0^*, r^*)$.

Remark that, as a consequence of this fact, we get that the measure of $A^*$ is exactly $\alpha$, $|A^*| = \alpha$.

Now, to end the proof, consider a normalized extremal $u^*$ associated to an optimal hole $A^* = \Omega \setminus B(x_0^*, r^*)$. As $u^*$ vanishes on $A^*$, attains its maximum at $x_0^*$ and $\|\nabla u^*\|_{L^\infty(\Omega)} = 1/r^*$, $u^*$ restricted to every line that joins $x_0^*$ and $y \in \partial A^* \cap \Omega = \partial B(x_0^*, r^*) \cap \Omega$ is a linear function with slope $1/r^*$. Therefore, we conclude that $u^*(x) = C_{x_0^*, r^*}(x)$, as we wanted to prove. □

### 3. The best Sobolev constant

Now we consider the best constant for the usual Sobolev embedding $W^{1, \infty}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Like for the best Sobolev trace constant in the previous section we want to show that the cone with vertex at a point $x_0^*$ on the boundary that maximizes the radius such that $|\Omega \setminus B(x_0^*, r)| = \alpha$ is an extremal for the optimization problem of minimizing

$$S(\alpha) = \inf_{A \subset \Omega, |A| \geq \alpha} S_A.$$

We just sketch the arguments since they are completely analogous to the previous ones. Details are left to the reader.
The existence of extremals for $S_A$ and the existence of an optimal hole $A^*$ can be shown in a completely analogous way as in the previous section, see Lemma 2.1 and Theorem 2.1.

Next we have that if we consider a fixed hole $A \subset \Omega$ with $|A| = \alpha < |\Omega|$ and a corresponding extremal $u$, then there exists an extremal for $S_A$ of the form $C_{x_0,r}$, with $r = \text{dist}(x_0, A)$, $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$. This plays a key role in the proof of Theorem 1.2, and is the analogous to Theorem 2.2 with a similar proof.

Once this result is proved the proof of Theorem 1.2 follows by the same arguments as used in the proof of Theorem 1.1.

References


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