

## Mathematical Tripos Part II: Michaelmas Term 2015

### Numerical Analysis – Lecture 8

**Problem 2.23 (The advection equation)** A useful paradigm for hyperbolic PDEs is the *advection equation*

$$u_t = u_x, \quad 0 \leq x \leq 1, \quad t \geq 0, \quad (2.10)$$

where  $u = u(x, t)$ . It is given with the initial condition  $u(x, 0) = \varphi(x)$ ,  $x \in [0, 1]$  and (for simplicity) the boundary condition  $u(1, t) = \varphi(t + 1)$ . The exact solution of (2.10) is simply  $u(x, t) = \varphi(x + t)$ , a unilateral shift leftwards. This, however, does not mean that its numerical modelling is easy.

**Example 2.24 (Instability)** We commence by semidiscretizing  $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h} [u_{m+1}(t) - u_{m-1}(t)]$ , so coming to the ODE  $u'_m(t) = \frac{1}{2h} [u_{m+1}(t) - u_{m-1}(t)]$ . For the Euler method, the outcome is

$$u_m^{n+1} = u_m^n + \frac{1}{2}\mu(u_{m+1}^n - u_{m-1}^n), \quad m = 0 \dots M, \quad n \in \mathbb{Z}_+,$$

with  $u_0^n = 0$  for all  $n$ . In matrix form this reads

$$\mathbf{u}^{n+1} = A\mathbf{u}^n, \quad A = \begin{bmatrix} 1 & \frac{1}{2}\mu & & & \\ -\frac{1}{2}\mu & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \frac{1}{2}\mu \\ & & & -\frac{1}{2}\mu & 1 \end{bmatrix}.$$

The matrix  $A$  is normal, with the eigenvalues  $\lambda_\ell = 1 + i\mu \cos \ell\pi h$  (see Example 2.16), so that  $\|A\|^2 = 1 + \mu^2$ , hence instability for any  $\mu$ .

**Method 2.25 (Upwind method)** If we semidiscretize  $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{h} [u_{m+1}(t) - u_m(t)]$ , and solve the ODE again by Euler's method, then the result is

$$u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n), \quad m = 0 \dots M, \quad n \in \mathbb{Z}_+ \quad (2.11)$$

The local error is  $\mathcal{O}(k^2 + kh)$  which is  $\mathcal{O}(h^2)$  for a fixed  $\mu$ , hence convergence if the method is stable.

The eigenvalue analysis of stability does not apply here, since the matrix  $A$  in  $\mathbf{u}^{n+1} = A\mathbf{u}^n$  is no longer normal (see Example 2.17), so we do it directly (as in Lecture 5). We let the boundary condition at  $x = 1$  be zero and define  $\|\mathbf{u}^n\| = \max_m |u_m^n|$ . It follows at once from (2.11) that

$$\|\mathbf{u}^{n+1}\| = \max_m |u_m^{n+1}| \leq \max_m \{|1 - \mu| |u_m^n| + \mu |u_{m+1}^n|\} \leq (|1 - \mu| + \mu) \|\mathbf{u}^n\|, \quad n \in \mathbb{Z}_+.$$

Therefore,  $\mu \in (0, 1]$  means that  $\|\mathbf{u}^{n+1}\| \leq \|\mathbf{u}^n\| \leq \dots \leq \|\mathbf{u}^0\|$ , hence stability.

**Matlab demo:** Download the Matlab GUI for *Solving the Advection Equation, Upwinding and Stability* from <http://www.maths.cam.ac.uk/undergrad/course/na/ii/advection/advection.php> and solve the advection equation (2.10) with the different methods provided in the demonstration. Experience what can go wrong when "winding" in the wrong direction!

**Method 2.26 (The leapfrog method)** We semidiscretize (2.10) as  $\frac{\partial u_m(t)}{\partial x} \approx \frac{1}{2h} [u_{m+1}(t) - u_{m-1}(t)]$ , but now solve the ODE with the second-order *midpoint rule*

$$\mathbf{y}_{n+1} = \mathbf{y}_{n-1} + 2k\mathbf{f}(t_n, \mathbf{y}_n), \quad n \in \mathbb{Z}_+.$$

The outcome is the two-step *leapfrog* method

$$u_m^{n+1} = \mu(u_{m+1}^n - u_{m-1}^n) + u_m^{n-1}. \quad (2.12)$$

The error is now  $\mathcal{O}(k^3 + kh^2) = \mathcal{O}(h^3)$ . We analyse stability by the Fourier technique, assuming that we are solving a Cauchy problem. Thus, proceeding as before,

$$\hat{u}^{n+1}(\theta) = \mu(e^{i\theta} - e^{-i\theta})\hat{u}^n(\theta) + \hat{u}^{n-1}(\theta) \quad (2.13)$$

whence

$$\widehat{u}^{n+1}(\theta) - 2i\mu \sin \theta \widehat{u}^n(\theta) - \widehat{u}^{n-1}(\theta) = 0, \quad n \in \mathbb{Z}_+,$$

and our goal is to determine values of  $\mu$  such that  $|\widehat{u}^n(\theta)|$  is uniformly bounded for all  $n, \theta$ . This is a difference equation  $w_{n+1} + bw_n + cw_{n-1} = 0$  with the general solution  $w_n = c_1\lambda_1^n + c_2\lambda_2^n$ , where  $\lambda_1, \lambda_2$  are the roots of the characteristic equation  $\lambda^2 + b\lambda + c = 0$ , and  $c_1, c_2$  are constants, dependent on the initial values  $w_0$  and  $w_1$ . If  $\lambda_1 = \lambda_2$ , then solution is  $w_n = (c_1 + c_2n)\lambda^n$ . In our case, we obtain

$$\lambda_{1,2}(\theta) = i\mu \sin \theta \pm \sqrt{1 - \mu^2 \sin^2 \theta}.$$

Stability is equivalent to  $|\lambda_{1,2}(\theta)| \leq 1$  for all  $\theta$  and this is true if and only if  $\mu \leq 1$ .

**Problem 2.27 (Stability in the presence of boundaries)** It is easy to extend Fourier analysis for the Euler method  $u_m^{n+1} = u_m^n + \mu(u_{m+1}^n - u_m^n)$ , with the initial condition  $u(x, 0) = \phi(x)$ ,  $x \in [0, 1)$ , and zero boundary condition along  $x = 1$ . Consider the Cauchy problem for the advection equation with the initial condition  $u(x, 0) = \phi(x)$  for  $x \in [0, 1)$ , and  $u(x, 0) = 0$  otherwise (it isn't differentiable, but this is not much of a problem). Solving the Cauchy problem with Euler, we recover  $u^n$  that is identical to the solution obtained from the zero boundary condition. This justifies using Fourier analysis for the problem with a boundary, and we obtain

$$\widehat{u}^{n+1}(\theta) = H(\theta) \widehat{u}^n(\theta), \quad H(\theta) = (1 - \mu) + \mu e^{i\theta},$$

so that  $\max |H(\theta)| = |1 - \mu| + |\mu|$ , hence stability if and only if  $\mu \leq 1$ .

Unfortunately, this is no longer true for leapfrog. Closer examination reveals that we cannot use leapfrog at  $m = 0$ , since  $u_{-1}^n$  is unknown. The naive remedy, setting  $u_{-1}^n = 0$ , leads to instability, which propagates from the boundary inwards. We can recover stability letting, for example,  $u_0^{n+1} = u_1^n$  (the proof is *very* difficult).

**Matlab demo:** The Matlab GUI for *Stability of 1D PDEs* features different choices of boundary conditions. A brief description of those is given at the bottom of the page [http://www.maths.cam.ac.uk/undergrad/course/na/ii/pde\\_stability/pde\\_stability.php](http://www.maths.cam.ac.uk/undergrad/course/na/ii/pde_stability/pde_stability.php). Go and see how the solution to the diffusion- or wave equation changes when changing the boundary conditions. Do you face any stability problems in those cases?

**Problem 2.28 (The wave equation)** Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad x \in [0, 1], \quad t \geq 0,$$

given with initial (for  $u$  and  $u_t$ ) and boundary conditions. The usual approximation looks as follows

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu(u_{m+1}^n - 2u_m^n + u_{m-1}^n),$$

with the Courant number being now  $\mu = k^2/h^2$ .

To advance in time we have to pick up the numbers  $u_m^1 = u(x_m, k)$  (of course they should depend on the initial derivative  $u_t(x, 0)$ ). Euler's method provides the obvious choice  $u(x_m, k) = u(x_m, 0) + ku_t(x_m, 0)$ , but the following technique enjoys better accuracy. Specifically, we set  $u_m^1$  to the right-hand side of the formula

$$\begin{aligned} u(x_m, k) &\approx u(x_m, 0) + ku_t(x_m, 0) + \frac{1}{2}k^2 u_{tt}(x_m, 0) \\ &= u(x_m, 0) + ku_t(x_m, 0) + \frac{1}{2}k^2 u_{xx}(x_m, 0) \\ &\approx u_m^0 + \frac{1}{2}\mu(u_{m-1}^0 - 2u_m^0 + u_{m+1}^0) + ku_t(x_m, 0). \end{aligned}$$

The Fourier analysis (for Cauchy problem) provides

$$\widehat{u}^{n+1}(\theta) - 2\widehat{u}^n(\theta) + \widehat{u}^{n-1}(\theta) = -4\mu \sin^2 \frac{\theta}{2} \widehat{u}^n(\theta),$$

with the characteristic equation  $\lambda^2 - 2(1 - 2\mu \sin^2 \frac{\theta}{2})\lambda + 1 = 0$ . The product of the roots is one, therefore stability (that requires the moduli of both  $\lambda$  to be at most one) is equivalent to the roots being complex conjugate, so we require

$$(1 - 2\mu \sin^2 \frac{\theta}{2})^2 \leq 1.$$

This condition is achieved if and only if  $\mu = k^2/h^2 \leq 1$ .