

Mathematical Tripos Part II: Michaelmas Term 2015

Numerical Analysis – Lecture 13

Problem 3.13 (The Poisson equation) We consider the *Poisson equation*

$$\nabla^2 u = f, \quad -1 \leq x, y \leq 1, \quad (3.11)$$

where f is analytic and obeys the periodic boundary conditions

$$f(-1, y) = f(1, y), \quad -1 \leq y \leq 1, \quad f(x, -1) = f(x, 1), \quad -1 \leq x \leq 1.$$

Moreover, we add to (3.11) the following *periodic boundary conditions*

$$\begin{aligned} u(-1, y) &= u(1, y), & u_x(-1, y) &= u_x(1, y), & -1 \leq y \leq 1 \\ u(x, -1) &= u(x, 1), & u_y(x, -1) &= u_y(x, 1), & -1 \leq x \leq 1. \end{aligned} \quad (3.12)$$

With these boundary conditions alone a solution of (3.11) is only defined up to an additive constant. Hence, we add a *normalisation condition* to fix the constant:

$$\int_{-1}^1 \int_{-1}^1 u(x, y) \, dx \, dy = 0. \quad (3.13)$$

We have the spectrally convergent Fourier expansion

$$f(x, y) = \sum_{k, l = -\infty}^{\infty} \hat{f}_{k, l} e^{i\pi(kx + ly)}$$

and seek the Fourier expansion of u

$$u(x, y) = \sum_{k, l = -\infty}^{\infty} \hat{u}_{k, l} e^{i\pi(kx + ly)}.$$

Since

$$0 = \int_{-1}^1 \int_{-1}^1 u(x, y) \, dx \, dy = \sum_{k, l = -\infty}^{\infty} \hat{u}_{k, l} \int_{-1}^1 \int_{-1}^1 e^{i\pi(kx + ly)} \, dx \, dy = \hat{u}_{0, 0},$$

and

$$\nabla^2 u(x, y) = -\pi^2 \sum_{k, l = -\infty}^{\infty} (k^2 + l^2) \hat{u}_{k, l} e^{i\pi(kx + ly)},$$

together with (3.11), we have

$$\begin{cases} \hat{u}_{k, l} = -\frac{1}{(k^2 + l^2)\pi^2} \hat{f}_{k, l}, & k, l \in \mathbb{Z}, (k, l) \neq (0, 0) \\ \hat{u}_{0, 0} = 0. \end{cases}$$

Matlab demo: See *FFT with Spectral Methods* at http://www.maths.cam.ac.uk/undergrad/course/na/ii/poisson_equation/poisson_equation.php for an implementation of the above numerical method for solving (3.11). In order to compute a solution to Poisson's equation download the m-files from the bottom of the page and run `poisson_equation.m` with Matlab. You can change the default forcing term f by changing g from $\sin(x\pi)\sin(y\pi)$ to any other periodic function you are interested in.

Remark 3.14 Applying a spectral method to the Poisson equation is not representative for its application to other PDEs. The reason is the special structure of the Poisson equation. In fact, $\phi_{k, l} = e^{i\pi(kx + ly)}$ are the eigenfunctions of the Laplace operator with

$$\nabla^2 \phi_{k, l} = -\pi^2(k^2 + l^2)\phi_{k, l},$$

and they obey periodic boundary conditions.

Problem 3.15 (General second-order linear elliptic PDE) We consider the more general second-order linear elliptic PDE

$$\nabla^\top(a\nabla u) = f, \quad -1 \leq x, y \leq 1,$$

with $a > 0$, and a and f periodic. We again impose the periodic boundary conditions (3.12) and the normalisation condition (3.13). We can rewrite

$$\nabla^\top(a\nabla u) = \frac{\partial}{\partial x}(au_x) + \frac{\partial}{\partial y}(au_y) = a\nabla^2 u + a_x u_x + a_y u_y$$

and get

$$\begin{aligned} & -\pi^2 \left(\sum_{k,l=-\infty}^{\infty} \hat{a}_{k,l} e^{i\pi(kx+ly)} \right) \left(\sum_{k,l=-\infty}^{\infty} (k^2 + l^2) \hat{u}_{k,l} e^{i\pi(kx+ly)} \right) \\ & - \pi^2 \left(\sum_{k,l=-\infty}^{\infty} k \hat{a}_{k,l} e^{i\pi(kx+ly)} \right) \left(\sum_{k,l=-\infty}^{\infty} k \hat{u}_{k,l} e^{i\pi(kx+ly)} \right) \\ & - \pi^2 \left(\sum_{k,l=-\infty}^{\infty} l \hat{a}_{k,l} e^{i\pi(kx+ly)} \right) \left(\sum_{k,l=-\infty}^{\infty} l \hat{u}_{k,l} e^{i\pi(kx+ly)} \right) \\ & = \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l} e^{i\pi(kx+ly)}, \end{aligned}$$

where

$$a(x, y) = \sum_{k,l=-\infty}^{\infty} \hat{a}_{k,l} e^{i\pi(kx+ly)}, \quad f(x, y) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l} e^{i\pi(kx+ly)}, \quad u(x, y) = \sum_{k,l=-\infty}^{\infty} \hat{u}_{k,l} e^{i\pi(kx+ly)}$$

In the next steps we replace the products by convolutions (using the bivariate version of (3.4)), truncate the expansions to $-N/2 + 1 \leq k, l \leq N/2$ and impose the normalisation condition $\hat{u}_{0,0} = 0$. This results in a system of $N^2 - 1$ linear algebraic equations in the unknowns $\hat{u}_{k,l}$, $k, l = -N/2 + 1, \dots, N/2$, $(k, l) \neq (0, 0)$:

$$\begin{aligned} & -\pi^2 \sum_{m,n=-N/2+1}^{N/2} [\hat{a}_{k-m,l-n}(m^2 + n^2) \hat{u}_{m,n} + (k-m) \hat{a}_{k-m,l-n} m \hat{u}_{m,n} \\ & \qquad \qquad \qquad + (l-n) \hat{a}_{k-m,l-n} n \hat{u}_{m,n}] = \hat{f}_{k,l}. \end{aligned}$$

Matlab demo: See the online documentation for *Spectral Methods for the Poisson Equation* at http://www.maths.cam.ac.uk/undergrad/course/na/ii/poisson_equation/poisson_equation.php for an exemplar Matlab code of how this can be implemented.

Discussion 3.16 (Analyticity and periodicity) The fast convergence of spectral methods rests on two properties of the underlying problem: analyticity and periodicity. If one is not satisfied the rate of convergence in general drops to polynomial. However, to a certain extent, we can relax these two assumptions while still retaining the substantive advantages of Fourier expansions.

- *Relaxing analyticity:* In general, the speed of convergence of the truncated Fourier series of a function f depends on the smoothness of the function. In fact, the smoother the function the faster the truncated series converges, i.e., for $f \in C^p(-1, 1)$ we receive an $\mathcal{O}(N^{-p})$ order of convergence.

Spectral convergence can be recovered, once analyticity is replaced by the requirement that $f \in C^\infty(-1, 1)$, i.e., $f^{(m)}(x)$ exists for all $x \in (-1, 1)$ and $m = 0, 1, 2, \dots$. Consider, for instance, $f(x) = e^{-1/(1-x^2)}$. Then, $f \in C^\infty(-1, 1)$ but cannot be extended analytically because of essential singularities at ± 1 . Nevertheless, one can show that $|\hat{f}_n| \sim \mathcal{O}(e^{-cn^\alpha})$, where $c > 0$ and $\alpha \approx 0.44$. While this is slower than exponential convergence in the analytic case (cf. Remark 3.7), it is still faster than $\mathcal{O}(n^{-m})$ for any integer m and hence, we have spectral convergence.

- *Relaxing periodicity:* Disappointingly, periodicity is necessary for spectral convergence. Once this condition is dropped, we are back to the setting of Theorem 3.3, i.e., Fourier series converge as $\mathcal{O}(N^{-1})$ unless $f(-1) = f(1)$. One way around this is to change our set of basis functions, e.g., to Chebyshev polynomials.