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## Mathematical Tripos Part II: Michaelmas Term 2015

# Numerical Analysis – Lecture 15

**Remark 3.23 (Chebyshev expansion for the derivatives)** For an analytic function $u$, the coefficients of the Chebyshev expansion for its derivatives $\breve{u}_n^{(q)}$ are given by

$$\breve{u}_n^{(q)} = \frac{2}{c_n} \sum_{\substack{p=n+1 \\ n+p \text{ odd}}}^{\infty} p\, \breve{u}_p^{(q-1)}, \quad \forall q \geq 1,$$

where

$$c_n = \begin{cases} 2 & n = 0 \\ 1 & n \geq 1. \end{cases}$$

The above recursion for the expansion coefficients is based on Lemma 3.21. The case $q = 1$ is the topic of Exercise 19 on the Example Sheets.

**Method 3.24 (The spectral method for evolutionary PDEs)** We consider the problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), & x \in [-1,1],\ t \geq 0 \\ u(x,0) = g(x), & x \in [-1,1], \end{cases} \tag{3.18}$$

with appropriate boundary conditions on $\partial[-1,1]$ and where $\mathcal{L}$ is a linear operator, e.g., a differential operator. We want to solve this problem by the method of lines (SD), using a spectral method for the approximation of $u$ and its derivatives in the spatial variable $x$. Then, in a general spectral method, we seek solutions $\phi_N(x,t)$ with

$$\phi_N(x,t) = \sum_{n=-N/2+1}^{N/2} a_n(t)\, \varphi_n(x),$$

where $a_n(t)$ are the expansion coefficients and $\varphi_n$ are basis functions chosen according to the specific structure of (3.18), e.g., the *Fourier expansion* with $a_n(t) = \hat{u}_n(t)$, $\varphi_n(x) = e^{i\pi n x}$ for periodic boundary conditions and a polynomial expansion such as the *Chebyshev method* with $a_n(t) = \breve{u}_n(t)$, $\varphi_n(x) = T_n(x)$ for other than periodic boundary conditions. The spectral approximation in space results into a system of ODEs for the expansion coefficients $a_n(t)$:

$$a_n'(t) = \mathcal{L}_N a_n(t). \tag{3.19}$$

**Example 3.25 (The diffusion equation)** We consider the diffusion equation for a function $u = u(x,t)$

$$\begin{cases} u_t = u_{xx}, & -1 \leq x \leq 1,\ t \geq 0 \\ u(x,0) = g(x), & -1 \leq x \leq 1, \end{cases} \tag{3.20}$$

with periodic boundary conditions and normalisation condition, i.e.,

$$\begin{aligned} &u(x=-1,t) = u(x=1,t),\ u_x(x=-1,t) = u_x(x=1,t), && t \geq 0 \\ &\int_{-1}^{1} u(x,t)\, dx = 0, && t \geq 0. \end{aligned} \tag{3.21}$$

We approximate $u$ by its $N$-th order Fourier expansion in $x$

$$u(x,t) \approx \sum_{n=-N/2+1}^{N/2} \hat{u}_n(t)\, e^{i\pi n x}.$$

Then, with (3.20), each coefficient $\hat{u}_n$ fulfills the ODE

$$\hat{u}_n'(t) = -\pi^2 n^2 \hat{u}_n(t), \quad n = -N/2+1, \ldots, N/2, \tag{3.22}$$

whose exact solution is $\hat{u}_n(t) = e^{-\pi^2 n^2 t}\ \hat{g}_n$ for $n \neq 0$ and $\hat{u}_0(t) = 0$ (due to the normalisation condition in (3.21)). Again, the diffusion equation benefits from the special structure of the Laplacian. However, for further reference, let us apply a finite difference approximation to (3.22), e.g., we approximate the ODE with the Euler method:

$$\hat{\boldsymbol{u}}^{k+1} = (Id + \Delta t \hat{A})\hat{\boldsymbol{u}}^k,$$

where $\hat{A} = \mathrm{diag}(-\pi^2 n^2)$.

**Example 3.26 (The diffusion equation with non-constant coefficient)** We want to solve the diffusion equation with non-constant coefficient $a(x)$ for a function $u = u(x,t)$

$$\begin{cases} u_t = (a(x)u_x)_x, & -1 \leq x \leq 1,\ t \geq 0 \\ u(x,0) = g(x), & -1 \leq x \leq 1, \end{cases} \tag{3.23}$$

conditioned to (3.21). Approximating $u$ by its truncated Fourier expansion results in the following system of ODEs for the coefficients $\hat{u}_n$

$$\hat{u}_n'(t) = -\pi^2 \sum_{m=-N/2+1}^{N/2} mn\ \hat{a}_{n-m}\ \hat{u}_m(t), \quad n = -N/2+1, \ldots, N/2.$$

For the discretization in time we apply finite differences, i.e., can solve this system of ODEs with our favourite ODE solver, e.g., with the Euler method. The latter amounts to compute

$$\hat{u}_n^{k+1} = \hat{u}_n^k - \Delta t \pi^2 \sum_{m=-N/2+1}^{N/2} mn\ \hat{a}_{n-m}\ \hat{u}_m^k,$$

or in vector form

$$\hat{\boldsymbol{u}}^{k+1} = (I + \Delta t \hat{A})\hat{\boldsymbol{u}}^k,$$

where $\hat{A} = (\hat{a}_{m,n}) = (-\pi^2 mn\ \hat{a}_{n-m})$

**Analysis 3.27 (Stability Analysis)** We consider the following finite difference approximation for (3.19)

$$\boldsymbol{a}^{k+1} = A^*(\Delta t, \mathcal{L}_N)\ \boldsymbol{a}^k, \tag{3.24}$$

with stepsize $\Delta t$ and $\boldsymbol{a} = (a_n)$. As in the previous section, the fully discrete scheme is stable provided that $\|A^*(\Delta t, \mathcal{L}_N)\| \leq 1$. If $A^*$ is a normal matrix this requirement is equivalent to $\rho(A^*) \leq 1$.

*Fourier methods*: For the truncated Fourier expansion $\boldsymbol{a} = \hat{\boldsymbol{u}}$ and the eigenvalue spectra $\lambda_n'$ and $\lambda_n''$ of the matrix approximating the first and second derivatives of $u$ respectively are

$$\lambda_n' \in \{-i(N/2-1), \ldots, -i, 0, i, \ldots, N/2\}, \quad \lambda_n'' \in \{-(N/2-1)^2, \ldots, -1, 0, -1, \ldots, -N^2/4\}.$$

In particular, the maximum eigenvalue for the $m$-th order differentiation is $\max|\lambda^{(m)}| = \left(\frac{N}{2}\right)^m$. This means that $\Delta t$ in explicit approximations (3.24) for linear PDEs with constant coefficients (for which $A^*$ is just a diagonal matrix, hence normal) must scale like $N^{-m}$, where $m$ is the maximal order of differentiation.

**Remark 3.28 (Chebyshev methods for evolutionary problems)** In general, the boundary conditions for the considered PDEs have to be implemented in the Chebyshev expansion. If the boundary conditions are to be imposed exactly, either the basis functions have to be slightly modified, e.g., to $T_n(x) - 1$ instead of $T_n(x)$ for the boundary condition $u(1) = 0$, or we get additional conditions on the expansion coefficients $\breve{u}_n$ (cf. Exercise 19 from the Example Sheets). While the exact imposition is in general not a problem for the numerical treatment of elliptic PDEs, as soon as the boundary conditions depend on time we may run into serious stability issues. One way around this is the use of penalty methods in which the boundary conditions is added to the scheme later as a penalty term.

**Matlab demo:** See the online documentation *Using Chebyshev Spectral Methods* at `http://www.maths.cam.ac.uk/undergrad/course/na/ii/chebyshev/chebyshev.php` for a simple example of how boundary conditions can be installed.