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Mathematical Tripos Part III: Lent Term 2013/14

Image Processing - Variational and PDE Methods – Examples' Sheet 1

1. (a) Prove that every $u \in L^1_{loc}(\Omega)$ can be understood as a distribution via the definition

$$\langle u, \varphi \rangle := \int_{\mathrm{supp}(\varphi)} u_{\mathrm{supp}\varphi} \varphi \, dx, \quad \text{for } \varphi \in \mathcal{D}(\Omega).$$

   (b) Show further that if $u, v \in L^1_{loc}$ and

$$\langle u, \varphi \rangle = \langle v, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega),$$

   then $u = v$ a.e.

   (c) A distribution that corresponds to a locally integrable function is called *regular*, otherwise it is called *singular*. Can you give an example for a singular distribution?

2. Prove that for a sequence (u_n) which is Cauchy in $L^p(\Omega)$ for a $1 \leq p < \infty$, we have that $\langle u_n, \varphi \rangle$ converges in $\mathbb{R}$ for every $\varphi \in \mathcal{D}(\Omega)$.

3. (a) Let $1 < p, q < \infty$. Prove that functions $u \in W^{p,q}(0,1)$ are Hölder continuous. What does this imply for a function $u \in W^{p,q}((0,1)^2)$?

   (b) *Non-examinable:* Prove further that for $1 \leq p \leq \infty$ a function $u \in W^{1,p}(0,1)$ agrees with a continuous function $\mathcal{L}^1$-almost everywhere.

4. For every $u \in L^1_{loc}(\Omega)$ the total variation of u is given by

$$TV(u) = |Du|(\Omega) = \sup \left\{ -\int_\Omega u \mathrm{div}\varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^2), \ |\varphi(x)| \leq 1 \ \forall x \in \Omega \right\}.$$

   For a function u with $TV(u) < \infty$ prove (by using Riesz representation theorem or otherwise) that the distributional derivative Du is a finite Radon measure. Show further that in this case $TV(u) \geq 0$.

5. Some special forms of the total variation:

   (a) Let $u \in W^{1,1}(\Omega)$. Prove that in this case $|Du|(\Omega) = \|Du\|_{L^1(\Omega)}$.

   (b) Let $\Omega \subset \mathbb{R}^2$, and let $B_r = B(0,r) \subset \Omega$ denote the disc centred at the origin with radius $r > 0$. Define χ_{B_r} the indicator function of B_r and compute its total variation.

   (c) Let $u \in L^q(\Omega)$ be a piecewise smooth function. More precisely, we assume that there exists $\Omega_1, \ldots, \Omega_K$ pairwise disjoint and bounded domains with Lipschitz boundary, such that $\mathrm{cl}(\Omega) = \bigcup_k \mathrm{cl}(\Omega_k)$ and $u^k := u|_{\Omega_k} \in C^1(\mathrm{cl}(\Omega_k))$. Prove that the total variation of u is given by

$$TV(u) = \|(Du)_{L^1}\|_1 + \sum_{l<k} \int_{\Gamma_{l,k}} |u^l - u^k| \, d\mathcal{H}^1,$$

   where $(Du)_{L^1}$ is the absolutely integrable part of Du and $\Gamma_{l,k} = \mathrm{cl}(\Omega_l) \cap \mathrm{cl}(\Omega_k) \cap \Omega$.

6. Show that $|Du|(\Omega)$ is a semi-norm in $BV(\Omega)$ and that $BV(\Omega)$ is a Banach space with norm $\|u\|_{BV} = \|u\|_1 + |Du|(\Omega)$.

7. Let X^* be the dual space of a separable normed space. Show that the functional $F : X^* \to \mathbb{R}$ that is bounded from below, coercive and weak* sequentially lower semi continuous has a minimiser in X^*.

8. Let Ω be a rectangular domain in $\mathbb{R}^2$ and $g \in L^2(\Omega)$. Prove that the ROF-minimiser u, that is

$$u = \mathrm{argmin}_{u \in L^2(\Omega)} \left\{ \alpha |Du|(\Omega) + \frac{1}{2} \|u - g\|_2^2 \right\},$$

   has to be an element in $BV(\Omega)$. Having Rellich's compactness theorem in mind, can you come up with another proof for the existence of the ROF-minimiser?

9. Prove that for every $\alpha > 0$ and a ROF-minimiser $u = u_\alpha$ we have

$$\int_\Omega g \, dx = \int_\Omega u_\alpha \, dx.$$

   Show further that for $\alpha \to +\infty$ the minimisers u_α converge in L^1 to the average of g.

10. Let X be a Banach space and $\mathcal{J}_1 : X \to \mathbb{R}$, $\mathcal{J}_2 : X \to \mathbb{R}$ two functionals. Prove that then

   (a) If $\mathcal{J}_1$ and $\mathcal{J}_2$ are weak l.s.c., $\alpha \geq 0$, then $\alpha\mathcal{J}_i$ and $\mathcal{J}_1 + \mathcal{J}_2$ are weak l.s.c.

   (b) If $\mathcal{J}_1$ is weak l.s.c. and $\phi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and l.s.c., then $\phi \circ \mathcal{J}_1$ is weak l.s.c.

   (c) If $\mathcal{J}_1$ is weak l.s.c. then for a Banach space Y and weak sequentially continuous $\Phi : \mathcal{Y} \to X$ it follows that $\mathcal{J}_1 \circ \Phi$ is weak l.s.c.

   (d) For any (non-empty) family of weak l.s.c. functionals $\mathcal{J}_i : X \to \mathbb{R}$, $i \in I$ we have that $\sup_{i\in I} \mathcal{J}_i$ is weak l.s.c.

   (e) Let $\phi : K \to \mathbb{R}$ be l.s.c., where K is either $\mathbb{R}$ or $\mathbb{C}$, and $x^* \in X^*$. Then the functional

$$L_{x^*,\phi} = \phi \circ \langle x^*, \cdot\rangle_{X^* \times X},$$

   is weak l.s.c. in X.

   Collect what you have proven (as appropriate) and show that $\phi(\|u\|_X)$ is weak l.s.c. for any $\phi : [0, \infty) \to \mathbb{R}$ that is monotonically increasing and l.s.c.

11. For a given noisy image $g \in L^2(\Omega)$ and rectangular image domain $\Omega \subset \mathbb{R}^2$ we consider the following variational problem

$$u_\alpha = \text{argmin}_{u\in L^2(\Omega)} \left\{\alpha\|Du\|_2^2 + \|u - g\|_2^2\right\},$$

   for $\alpha > 0$. Prove that there exists a unique minimiser u_α to the above problem. Show further, that under the additional assumption that $L \leq g \leq R$ a.e. in Ω the minimiser u_α of the above problem also fulfils $L \leq u_\alpha \leq R$ a.e. in Ω.

12. Let $g \in L^q(\Omega')$, $\infty > q > 1$, be a noisy and blurry image with blurring kernel $k \in L^1(\Omega_0)$ and $\int_\Omega k\, dx$=1. We want to reconstruct a denoised and deblurred image u on a rectangular domain Ω for which $\Omega' - \Omega_0 \subset \Omega$ under the assumption that

$$\text{for } x \in \Omega' : \quad (u * k)(x) = \int_\Omega u(x - y)k(y)\, dy.$$

   Prove that the variational problem

$$\min_{u\in L^q(\Omega)} \frac{\alpha}{p}\|Du\|_p^p + \frac{1}{q}\int_{\Omega'} |u * k - g|^q\, dx,$$

   attains a unique minimiser for $1 < p \leq q < \infty$, $g \in L^q(\Omega')$ and $\alpha > 0$. In the case $p = q = 2$ derive the Euler-Lagrange equation with appropriate boundary conditions that corresponds to the above minimisation problem. What is the relation between solutions of the Euler-Lagrange problem and the minimiser of the variational problem?