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Mathematical Tripos Part III: Lent Term 2013/14

Image Processing - Variational and PDE Methods – Examples' Sheet 2

1. Let the functional $\mathcal{J} : X \to \mathbb{R}$ be convex on a real Banach space X. Prove that:
   (a) For $p \in \partial\mathcal{J}(u)$ and $q \in \partial J(u)$ we have $tp + (1-t)q \in \partial\mathcal{J}(u)$ for all $t \in [0,1]$.
   (b) Let $\mathcal{J}$ be l.s.c. and consider a sequence $((u_n, p_n))$ in $X \times X^*$ with $p_n \in \partial\mathcal{J}(u_n)$, $u_n \to u$ and $p_n \overset{*}{\rightharpoonup} p$. Then $p \in \partial\mathcal{J}(u)$.
   (c) For $u \in X$ the set $\partial\mathcal{J}(u)$ is weak* sequentially closed, that is:
   $$\text{For } p_n \overset{*}{\rightharpoonup} p, \text{ we have that } p \in \partial\mathcal{J}(u).$$

2. For an inpainting domain $D \subset \Omega$ and a given noise free image $g|_{\Omega\setminus D}$, a regularised version of the total variation model for image inpainting is the minimisation of the modified energy
   $$\mathcal{J}_\epsilon(u) = \int_D \sqrt{\epsilon + |Du|^2}\, dx, \qquad \text{with } u = g \text{ on } \partial D.$$
   Since this Dirichlet problem might not be solvable for general inpainting domains D, we consider a weaker version for the noise-free case:
   $$\min\left\{\int_D \sqrt{\epsilon + |Du|^2}\, dx + \frac{\mu}{2}\int_{\partial D} (u-g)^2\, d\mathcal{H}^1\right\},$$
   where $\mu \gg 1$ is a large positive weight. Prove the existence of a minimiser for the above problem. Is this minimiser unique?

3. Let γ be a twice continuously differentiable curve in the plane. Check that the modulus of the curvature of $\gamma = \gamma(s)$ with arc-length s is given by $|\kappa(s)| = |\gamma''(s)|$. In particular prove that the latter is independent of the parametrisation, i.e. $\gamma(s(t)) = \gamma(s(\tilde{t}))$.

4. Let $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ twice continuously differentiable in a neighbourhood of a point (x_0, y_0) with $u(x_0, y_0) = 0$. Moreover, we assume that the zero level line $\Gamma_0 = \{u = 0\}$ is locally described by the maps $x, y : (-\epsilon, \epsilon) \to \mathbb{R}$ which are twice continuously differentiable and parameterised by arc length, that is $(x(0), y(0)) = (x_0, y_0)$, $|x'(s)|^2 + |y'(s)|^2 = 1$ and $u(x(s), y(s)) = 0$ for all $s \in (-\epsilon, \epsilon)$. Prove that, with
   $$Du(x(s), y(s)) \neq \mathbf{0}, \text{ and } \operatorname{div}\left(\frac{Du(x(s), y(s))}{|Du(x(s), y(s))|}\right) = \kappa,$$
   for a $\kappa \in \mathbb{R}$ and all $s \in (-\epsilon, \epsilon)$, there exists a $\varphi_0 \in \mathbb{R}$, such that (x, y) can be written as
   $$x(s) = x_0 + \sin(\kappa s + \varphi_0), \quad y(s) = \varphi_0 + \cos(\kappa s + \varphi_0),$$
   for all $s \in (-\epsilon, \epsilon)$. In particular, (x, y) parametrises a line segment or a segment of a circle with curvature κ.

5. (a) Let $\phi \in C_c^1(\mathbb{R}, [0, \infty))$ and define
   $$R(u) = \int_\Omega \phi(\kappa) \cdot |\nabla u|\, dx,$$
   for $u \in W^{2,1}(\Omega)$. Derive the first variation of R over the set of $C_c^\infty(\Omega)$ in terms of
   $$\nabla_u R = -\operatorname{div}\vec{V},$$
   where $\vec{V}$ is a vector field in $\mathbb{R}^2$. Write $\vec{V}$ as the sum of two components that act normal and tangential to the level lines of u, respectively. Use your findings to give an interpretation of Euler's elastica inpainting in terms of the two mechanisms, transport and diffusion.
   (b) Now, let $\phi \in C^1(\mathbb{R}, (0, \infty))$. Taking now the first variation of R over $C^\infty(\Omega)$, what are the boundary conditions that you get along $\partial\Omega$?
   (c) Now consider
   $$R(u) = \int_\Omega \phi(\Delta u)\, dx,$$
   where $\phi'(s) = \arctan(s/\delta)$, $\delta > 0$. Derive the first variation of R and give an interpretation of the inpainting dynamics of this regulariser in terms of transport and diffusion. What is the role of δ?

6. We use the following definition.

Definition 2.1 (Directional Hessian) *For $u : \Omega \to \mathbb{R}$ let D^2u be (as usual) the Hessian of u. Then, we can specify second-order derivatives in a specific direction by assigning the following quadratic form to $A = D^2u$*

$$Au\,(x,y) = \sum_{i,j} A_{i,j} x_i y_j, \quad A \in \mathbb{R}^{2\times 2}, x, y \in \mathbb{R}^2.$$

Like that, we get the second partial derivatives in the direction of the gradient of u as $u_{\eta\eta} = D^2u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)$, and the second partial derivatives in the direction tangent to the level lines as $u_{\xi\xi} = D^2u\left(\frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|}\right)$. Of course, we can have the mixed case as well, which gives $u_{\xi\eta} = D^2u\left(\frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)$.

Prove that the problem $\Delta u = 0$ in $B(0,r) \setminus \{0\}$, $u = 0$ on $\partial B(0,r)$ and $u(0) = 1$ does not have a solution. Prove that the same result holds for any linear combination of the form

$$\alpha D^2u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) + \beta D^2u\left(\frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|}\right), \quad \alpha, \beta > 0.$$

In contrary to the diffusion equation, prove that the so-called AMLE (absolutely minimising Lipschitz extension) equation does indeed have a solution to the above problem, that is

$$\begin{aligned} &D^2u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) = 0, && \text{in } B(0,r) \setminus \{0\} \\ &u = 0, && \text{on } \partial B(0,r) \\ &u(0) = 1, && \end{aligned}$$

obtains the solution $u(x) = 1 - |x|$.

7. (a) Verify the identity

$$|\nabla u| \cdot \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = D^2u\left(\frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|}\right).$$

   This gives the interpretation of the mean curvature of a function u as a directional diffusion u, the direction given by the orthogonal to the gradient of the function.

   (b) Now assume that $u(x_0)$ equals the median value of the set $\{u(y),\ y \in B_{x_0}(h)\}$, where $B_{x_0}(h)$ is a disc with radius h and centre x_0. Prove that, as $h \to 0$ we have

$$\operatorname{div}\left(\frac{\nabla u(x_0)}{|\nabla u(x_0)|}\right) = 0$$

   (c) Moreover, prove by Taylor expansion that the directional diffusion equation

$$D^2u\,(\nabla u, \nabla u) = 0.$$

   is the asymptotic limit for $h \to 0$ of

$$u(x) = \frac{1}{2}\left(u(x + h\nabla u) + u(x - h\nabla u)\right) + o(h^2).$$

   What is the geometric difference between the directional diffusion in (b) and (c)? What do they do to an image function u?

8. Let $\tilde{S} := (0,1)$, and for $i = 1, 2, \ldots$

$$S_i := \bigcup_{k=0}^{2^{i-1}-1} \left(k\frac{1}{2^{i-1}}, k\frac{1}{2^{i-1}} + \frac{1}{2^i}\right),$$

   i.e. $S_1 = (0, 0.5)$, $S_2 = (0, 0.25) \cup (0.5, 0.75)$, $S_3 = (0, 0.125) \cup (0.25, 0.375) \cup (0.5, 0.625) \cup (0.75, 0.875), \ldots$. Verify, that the sequence S_i converges in Hausdorff measure to $\tilde{S}$ for $i \to \infty$ while $\mathcal{H}^1(S_i) = 0.5$ for all $i \geq 1$ and $\mathcal{H}^1(\tilde{S}) = 1$. What does this fact tell you about the semicontinuity of $\mathcal{H}^1$ with respect to the Hausdorff topology?