Dr C.-B. Schönlieb

Mathematical Tripos Part II: Michaelmas Term 2015

Numerical Analysis – Examples' Sheet 2

11. Let $a(x) > 0$, $x \in [0,1]$, be a given smooth function. We solve the diffusion equation with variable diffusion coefficient, $u_t = (au_x)_x$, given with an initial condition for $t = 0$ and boundary conditions at $x = 0$ and $x = 1$, $t \geq 0$, with the finite-difference method

$$u_m^{n+1} = u_m^n + \mu \left[a_{m-1/2} u_{m-1}^n - (a_{m-1/2} + a_{m+1/2}) u_m^n + a_{m+1/2} u_{m+1}^n \right],$$

where $a_s = a(sh)$, $\mu = \frac{\Delta t}{(\Delta x)^2}$, $n \geq 0$, $1 \leq m \leq M$ and $h = \Delta x = \frac{1}{M+1}$. Prove that the local error is $\mathcal{O}(h^4)$. Then, justifying carefully every step of your analysis, show (e.g. by using the eigenvalue technique) that the method is stable for all $0 < \mu < \frac{1}{2a_{max}}$, where $a_{max} = \max_{x \in [0,1]} a(x)$. [*Hint: In the second half, use Gershgorin theorem to show that the matrix A occuring in the relation $u^{n+1} = Au^n$ satisfies $\rho(A) \leq 1$.*]

12. Apply the Fourier stability test to the difference equation

$$u_m^{n+1} = \frac{1}{2}(2 - 5\mu + 6\mu^2) u_m^n + \frac{2}{3}\mu(2 - 3\mu)(u_{m-1}^n + u_{m+1}^n) - \frac{1}{12}\mu(1 - 6\mu)(u_{m-2}^n + u_{m+2}^n),$$

where $m \in \mathbb{Z}$. Deduce that the test is satisfied if and only if $0 \leq \mu \leq \frac{2}{3}$.

13. A square grid is drawn on the region $\{(x,t) : 0 \leq x \leq 1,\ t \geq 0\}$ in $\mathbb{R}^2$, the grid points being $(m\Delta x, n\Delta x)$, $0 \leq m \leq M+1$, $n = 0, 1, 2, \ldots$, where $\Delta x = \frac{1}{M+1}$ and M is odd. Let $u(x,t)$ be an exact solution of the wave equation $u_{tt} = u_{xx}$ and let the boundary values $u(x,0)$, $0 \leq x \leq 1$, $u(0,t)$, $t > 0$, and $u(1,t)$, $t > 0$, be given. Further, an approximation to $\partial u/\partial t$ at $t = 0$ allows each of the function values $u(m\Delta x, \Delta x)$, $m = 1, 2, \ldots, M$, to be estimated to accuracy ϵ. Then, the difference equation

$$u_m^{n+1} = u_{m+1}^n + u_{m-1}^n - u_m^{n-1}$$

is applied to estimate u at the remaining grid points. Prove that all of the moduli of the errors $|u_m^n - u(m\Delta x, n\Delta x)|$ are bounded above by $\frac{1}{2}\epsilon M$, even when n is very large. [*Hint: Verify that the local error is zero. For $n = 1$ and $1 \leq m \leq M$, let the error in $u(m\Delta x, \Delta x)$ be $\delta_{mk}\epsilon$, where δ_{mk} is the Kronecker delta and where k is an arbitrary integer in $(1, 2, \ldots, M)$. Draw a diagram that shows the contribution from this error to u_m^n for every m and $n > 1$.*]

Matlab demo: Download the Matlab GUI for *Stability of 1D PDEs* at `http://www.maths.cam.ac.uk/undergrad/course/na/ii/pde_stability/pde_stability.php`. Review the stability condition from the lectures Problem 2.28 and test its sharpness empirically using the GUI.

14. A rectangular grid is drawn on $\mathbb{R}^2$, with grid spacing Δx in the x-direction and Δt in the t-direction. Let the difference equation

$$\begin{aligned} &u_m^{n+1} - 2u_m^n + u_m^{n-1} \\ &\quad = \mu \left[a \left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1} \right) + b \left(u_{m-1}^n - 2u_m^n + u_{m+1}^n \right) + c \left(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1} \right) \right], \end{aligned}$$

where $\mu = \frac{(\Delta t)^2}{(\Delta x)^2}$, be used to approximate solutions of the wave equation $u_{tt} = u_{xx}$. Deduce that, with constant μ, the local error is $\mathcal{O}((\Delta x)^4)$ if and only if the parameters a, b and c satisfy $a = c$ and $a + b + c = 1$. Show also that, if these conditions hold, then the Fourier stability condition is achieved for all values of μ if and only if the parameters also satisfy $|b| \leq 2a$. [*Hint: In the second half, the roots of the characteristic equation satisfy $x_1 x_2 = 1$. Then, $|x_1|, |x_2| \leq 1$ if $D \leq 0$, where D is the discriminant of the equation.*]

15. For a given analytic function f we consider its truncated Fourier approximation on the interval $[-1, 1]$, i.e.,

$$f(x) \approx \phi_N(x) = \sum_{n=-N/2+1}^{N/2} \hat{f}_n e^{i\pi n x}, \quad \text{where } \hat{f}_n = \frac{1}{2} \int_{-1}^{1} f(\tau) e^{-i\pi n \tau}\, d\tau, \quad n \in \mathbb{Z}.$$

Prove that for every $s = 1, 2, \ldots$ it is true for every $n \in \mathbb{Z} \setminus \{0\}$ that

$$\hat{f}_n = \frac{(-1)^{n-1}}{2} \sum_{m=0}^{s-1} \frac{1}{(\pi i n)^{m+1}} \left[f^{(m)}(1) - f^{(m)}(-1) \right] + \frac{1}{(\pi i n)^s} \widehat{f^{(s)}}_n.$$

16. Unless f is analytic, the rate of decay of its Fourier harmonics can be very slow, certainly slower than $\mathcal{O}(N^{-1})$. To explore this, let $f(x) = |x|^{-1/2}$. Prove that $\hat{f}_n = g(-n) + g(n)$, where $g(n) = \int_0^1 e^{i\pi n \tau^2}\, d\tau$. Moreover, with the error function erf defined as the integral

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2}\, d\tau, \quad z \in \mathbb{C}.$$

show that its Fourier coefficients are

$$\hat{f}_n = \frac{\operatorname{erf}(\sqrt{i\pi n})}{2\sqrt{in}} + \frac{\operatorname{erf}(\sqrt{-i\pi n})}{2\sqrt{-in}},$$

and asymptotically for $|n| \gg 1$ we have $\hat{f}_n = \mathcal{O}(n^{-1/2})$. [*Hint: For the last identity use without proof the asymptotic estimate* $\operatorname{erf}(\sqrt{ix}) = 1 + \mathcal{O}(x^{-1})$ *for* $x \in \mathbb{R}$, $|x| \gg 1$.]

17. Consider the solution of the two-point boundary value problem

$$(2 - \cos \pi x)u'' + u = 1, \quad -1 \le x \le 1, \quad u(-1) = u(1),$$

using the spectral method. Plugging the Fourier expansion of u into this differential equation, show that the $\hat{u}_n$ obey a three-term recurrence relation. Calculate $\hat{u}_0$ separately and using the fact that $\hat{u}_{-n} = \hat{u}_n$ (why?), prove further that the computation of $\hat{u}_n$ for $-N/2+1 \le n \le N/2$ (assuming that $\hat{u}_n = 0$ outside this range of n) reduces to the solution of an $(N/2) \times (N/2)$ tridiagonal system of algebraic equations.

18. Set

$$a(x) = \sum_{n=-\infty}^{\infty} \hat{a}_n\, e^{i\pi n x}, \tag{2.1}$$

the Fourier expansion of a. Explain why a is periodic with period 2. Further, let $\tilde{n}$ denote some selected value of n. Evaluate $\frac{1}{2}\int_{-1}^{1} a(x)\, e^{-i\pi\tilde{n}x}\, dx$ with $a(x)$ given by (2.1). Doing so, you have just computed the Fourier coefficient $\hat{a}_{\tilde{n}}$. Now choose $a(x) = \cos \pi x$ and compute its corresponding Fourier coefficients. With this, derive an explicit expression for the coefficients in the N-term truncated Fourier approximation of the solution u of

$$\begin{cases} ((\cos \pi x + 2)u_x)_x = \sin \pi x, \ x \in [-1, 1] \\ \text{periodic boundary conditions and normalisation condition } \displaystyle\int_{-1}^{1} u(x)\, dx = 0. \end{cases}$$

19. Let u be an analytic function in $[-1, 1]$ that can be extended analytically into the complex plane and possesses a Chebyshev expansion $u = \sum_{n=0}^{\infty} \breve{u}_n T_n$. Express u' in an explicit form as a Chebyshev expansion.

20. The two-point ODE $u'' + u = 1$, $u(-1) = u(1) = 0$, is solved by a Chebyshev method.
    (a) Show that the odd coefficients are zero and that $u(x) = \sum_{n=0}^{\infty} \breve{u}_{2n} T_{2n}(x)$. Express the boundary conditions as a linear condition of the coefficients $\breve{u}_{2n}$.
    (b) Express the differential equation as an infinite set of linear algebraic equations in the coefficients $\breve{u}_{2n}$.
    (c) Discuss how to truncate the linear system, keeping in mind the exponential convergence of the method and the floating-point precision of your computer.
    (d) While $u(-1) = u(1)$ we cannot expect a standard spectral method to converge at spectral speed. Why?

    Matlab demo: Compare your conclusions with the online documentation for solving this ODE at `http://www.maths.cam.ac.uk/undergrad/course/na/ii/chebyshev/chebyshev.php`. How are the boundary conditions enforced in practice?