Standard Model | Mathematical Tripos, Part III
Dr C E Thomas | Lent Term 2019

Example Sheet 1

[You may submit questions 2 and 4 to be marked.]

1. The four-dimensional 4×4 Dirac matrices are defined uniquely up to an equivalence by $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}1$, with 1 the unit matrix. We may also require that if $\gamma^\mu=(\gamma^0,\boldsymbol{\gamma})$ then $\gamma^{\mu\dagger}=(\gamma^0,-\boldsymbol{\gamma})$. If $[X,\gamma^\mu]=0$ for all μ then $X\propto 1$ and if γ^μ,γ'^μ both obey the Dirac algebra then $\gamma'^\mu=S\gamma^\mu S^{-1}$ for some S. Define the charge conjugation matrix C by $C\gamma^{\mu\,T}C^{-1}=-\gamma^\mu$, where T denotes the matrix transpose. Show that $[C^TC^{-1},\gamma^\mu]=0$ and hence that $C^T=cC,\ c=\pm 1$. Derive the results
$$\big(\gamma^\mu C\big)^T=-\,c\gamma^\mu C\,,\quad \big(\gamma^5 C\big)^T=c\gamma^5 C$$
$$\big(\gamma^\mu\gamma^5 C\big)^T=c\gamma^\mu\gamma^5 C\,,\quad \big([\gamma^\mu,\gamma^\nu]C\big)^T=-c[\gamma^\mu,\gamma^\nu]C\,.$$
Hence, since there are 6 independent antisymmetric and 10 symmetric 4×4 matrices, show that we must take $c=-1$. [Hint: the set of matrices $\{I,\gamma^\mu,\gamma^5,\gamma^\mu\gamma^5,[\gamma^\mu,\gamma^\nu]\}$ form a basis for 4×4 matrices.]

   Using the assumed Hermiticity properties of the Dirac matrices, show $[\gamma^\mu,CC^\dagger]=0$ so that we may take $CC^\dagger=1$.

   The matrix B is defined by $B^{-1}\gamma^{\mu *}B=(\gamma^0,-\boldsymbol{\gamma})$. Show that $B^{-1}\gamma^{5*}B=\gamma^5$. With the assumed form for $\gamma^{\mu\dagger}$ verify that we may take $B^{-1}=\pm\gamma^5 C$.

   *Generalise the above argument for finding c to $2n$ dimensions when the Dirac matrices are $2^n\times 2^n$ and we may take as a linearly independent basis 1 and $\gamma^{\mu_1\cdots\mu_r}=\gamma^{[\mu_1}\ldots\gamma^{\mu_r]}$, where [ ] denotes antisymmetrisation of indices, for $r=1,\ldots 2n$ ($\gamma^{\mu_1\cdots\mu_r}$ has $\binom{2n}{r}$ independent components). Show that $C(\gamma^{\mu_1\cdots\mu_r})^TC^{-1}=(-1)^{\frac12 r(r+1)}\gamma^{\mu_1\cdots\mu_r}$ and hence $c=(-1)^{\frac12 n(n+1)}$. Generalise γ^5 by taking $\hat\gamma=-i^{n-1}\gamma^0\gamma^1\ldots\gamma^{2n-1}$ and show that $\hat\gamma$ is Hermitian and $\hat\gamma^2=1$. Show that
$$\psi^c=C\bar\psi^T\,,\quad \psi'=\hat\gamma\psi\quad\Rightarrow\quad \psi=-c\,C\bar\psi^{c\,T}\,,\quad \psi'^c=-(-1)^n\hat\gamma\psi^c\,.$$
In what dimensions is it possible to have Majorana-Weyl spinors, so that $\psi^c=\pm\psi'=\psi$?

2. A Dirac quantum field transforms under parity so that
$$\hat P\psi(x)\hat P^{-1}=\gamma^0\psi(x_P)\,,\quad x_P^\mu=(x^0,-\mathbf{x})\,,$$
and has an interaction with a scalar field $\phi(x)$
$$\mathcal{L}_I(x)=g\,\bar\psi(x)\psi(x)\phi(x)+g'\bar\psi(x)i\gamma^5\psi(x)\phi(x)\,.$$
Obtain the necessary form for $\hat P\phi(x)\hat P^{-1}$ to ensure that the theory is invariant under parity if $g'=0$. What are the transformation properties of $\phi(x)$ for parity invariance when instead $g=0$? Can parity be conserved in a theory if both g,g' are non zero?

   How does the axial current $j_5^\mu(x)=\bar\psi(x)\gamma^\mu\gamma^5\psi(x)$ transform under parity?

3. For a free operator Dirac field $\hat{\psi}(x)$ assume $\hat{\psi}(x) = \sum_r a_r\,\psi_r(x)$ where $\{\psi_r(x)\}$ forms a basis for solutions of the Dirac equation and a_r are operators. Explain why a basis may be chosen so that $B^{-1}\psi_r^*(x) = \psi_{r'}(x_T)$ where $x_T^\mu = (-x^0, \mathbf{x})$ and $B^{-1}\gamma^{\mu *}B = (\gamma^0, -\boldsymbol{\gamma})$. Assume that the time-reversal operator is defined so that $\hat{T}a_r\hat{T}^{-1} = a_{r'}$. What is $\hat{T}\hat{\psi}(x)\hat{T}^{-1}$?

4. Under charge conjugation and time reversal a Dirac field ψ transforms as

$$\hat{C}\psi(x)\hat{C}^{-1} = C\bar{\psi}^T(x)\,, \qquad \hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)\,, \quad x_T^\mu = (-x^0, \mathbf{x})\,.$$

with $\hat{C}$, $\hat{T}$ the unitary, anti-unitary operators implementing these operations (recall that if $\hat{T}|\phi\rangle = |\phi_T\rangle$ then $\langle\phi'|\phi\rangle = \langle\phi_T|\phi'_T\rangle$). The matrices C, B are defined in question 1 and note that $C^\dagger C = B^\dagger B = 1$. Show that, if X is a matrix acting on Dirac spinors,

$$\hat{C}\,\bar{\psi}(x)X\psi(x)\hat{C}^{-1} = \bar{\psi}(x)X_C\psi(x)\,, \quad \hat{T}\,\bar{\psi}(x)X\psi(x)\hat{T}^{-1} = \bar{\psi}(x_T)X_T\psi(x_T)\,,$$

where $X_C = CX^TC^{-1}$ (take ψ and $\bar{\psi}$ to anti-commute) and $X_T = B^{-1}X^*B$. Hence determine the transformation properties under charge conjugation and time reversal of

$$\bar{\psi}(x)\psi(x)\,, \qquad \bar{\psi}(x)i\gamma^5\psi(x)\,, \qquad \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\,.$$

If $|\pi\rangle$ is a boson with momentum p and $\langle 0|\bar{\psi}(0)i\gamma^5\psi(0)|\pi(p)\rangle \neq 0$ show that, in a theory in which parity and charge conjugation are conserved, then the boson must have negative intrinsic parity and also positive charge-conjugation parity.

5. From Maxwell's equation $\partial_\nu F^{\mu\nu} = e\bar{\psi}\gamma^\mu\psi$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, derive the required transformation properties of $A_\mu(x)$ to ensure that Maxwell's equation is invariant under parity, charge conjugation and time reversal. Show that $\int d^4x\,\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}$ is odd under both parity and time reversal.

6. For a Dirac field ψ define $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$. Show that $\bar{\psi}_\pm\gamma^5 = \mp\bar{\psi}_\pm$. Let $\Psi_\pm = \begin{pmatrix}\psi_\pm \\ C\bar{\psi}_\mp^T\end{pmatrix}$ and show that then $\bar{\Psi}_\pm = \left(\bar{\psi}_\pm, -\psi_\mp^T C^{-1}\right)$. [Hint: it is easier to keep the new 2-dimensional "super-spin" space separate from the 4-dimensional spinor space of ψ.] A generalized Lorentz-invariant mass term can be written as

$$\mathcal{L}_m = \frac{1}{2}\Psi_+^T C^{-1}\mathcal{M}\Psi_+ \,-\, \frac{1}{2}\bar{\Psi}_+\mathcal{M}^* C\bar{\Psi}_+^T$$

where $\mathcal{M}$ is a symmetric 2×2 matrix which commutes with C and γ^μ. [The notation can be confusing but it is conventional. You can read the matrices more explicitly as $\mathbb{1}_4 \otimes \mathcal{M}$, $C \otimes \mathbb{1}_2$ and $\gamma^\mu \otimes \mathbb{1}_2$]. Verify that $\mathcal{L}_m^\dagger = \mathcal{L}_m$.

(i) If $\mathcal{M} = \begin{pmatrix}0 & m \\ m & 0\end{pmatrix}$ show that by absorbing any phase into $\psi_\pm$ we can take m real and positive, and that this reduces to the conventional Dirac mass term $\mathcal{L}_m = -m\bar{\psi}\psi$. Show that the kinetic term $\mathcal{L}_K = \bar{\psi}i\partial\!\!\!/\psi = \bar{\Psi}_+ i\partial\!\!\!/\Psi_+$. Regarding Ψ_+ and $\bar{\Psi}_+$ as independent and assuming $\mathcal{L} = \mathcal{L}_K + \mathcal{L}_m$, derive the equations

$$i\partial\!\!\!/\Psi_+ - \mathcal{M}^* C\bar{\Psi}_+^T = 0, \quad i\partial\!\!\!/ C\bar{\Psi}_+^T - \mathcal{M}\Psi_+ = 0$$

Hence show that the mass-squared eigenvalues are found by solving

$$\det\left(p^2 1 - \mathcal{M}^* \mathcal{M}\right) = 0\,.$$

(ii) Requiring $\hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)$ and $\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B^{-1}$, with $B = \pm(\gamma^5 C)^{-1} = \pm C^{-1}\gamma^5$ as in question 1, show that $\hat{T}\Psi_+(x)\hat{T}^{-1} = B\Psi_+(x_T)$. Hence demonstrate that $\mathcal{M}$ should be real in order to have $\hat{T}\mathcal{L}_m(x)\hat{T}^{-1} = \mathcal{L}_m(x_T)$.

(iii) If $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$ with m real and positive and $|M| \gg m$, show that the masses are approximately $|M|$ and $m^2/|M|$.

Comments/corrections to `C.E.Thomas@damtp.cam.ac.uk`.