Example Sheet 2

[You may submit questions 1.i and 3 to be marked.]

1. A field theory is described in terms of the elements of a complex $N \times N$ matrix M by a Lagrangian

$$\mathcal{L} = \mathrm{Tr}(\partial^\mu M^\dagger \partial_\mu M) - \tfrac{1}{2}\lambda\,\mathrm{Tr}(M^\dagger M M^\dagger M) - k\,\mathrm{Tr}(M^\dagger M)\,,$$

where Tr denotes the matrix trace and $\lambda > 0$.

(i) Show that this theory is invariant under the symmetry group $\big(U(N)\times U(N)\big)/U(1)$ for transformations given by $M \mapsto AMB^{-1}$ for $A, B \in U(N)$ and where the $U(1)$ corresponds to $A = B = e^{\mathrm{i}\theta}I$. [Note that if H is a subgroup of G then G/H is a group if H belongs to the centre of G, i.e. $hg = gh$ for all $h \in H$, $g \in G$.] Show that if $k < 0$ spontaneous symmetry breakdown occurs and that in the ground state ${M_0}^\dagger M_0 = v^2 I$ for some v. What is the unbroken symmetry group and how many Goldstone modes are there?

(ii) If $\mathcal{L} \to \mathcal{L} + \mathcal{L}'$ where

$$\mathcal{L}' = h\left(\det M + \det M^\dagger\right)\,,$$

what is the symmetry group and how many Goldstone modes are there now after spontaneous symmetry breakdown? [Assume the ground state still satisfies ${M_0}^\dagger M_0 = v^2 I$.]

[Note $U(N) = \big(SU(N)\times U(1)\big)/Z_N$ where Z_N is the finite group corresponding to the complex numbers $e^{2\pi \mathrm{i}k/N}$, $k = 0, \ldots N-1$, under multiplication.]

2. A field theory has 5 real scalar fields ϕ_a which are expressed in terms of a symmetric traceless 3×3 matrix $\Phi = \sum_{a=1}^{5} \phi_a t_a$ where t_a are a basis of symmetric traceless matrices with $\mathrm{Tr}(t_a t_b) = \delta_{ab}$. The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}(\partial^\mu \Phi \partial_\mu \Phi) - V(\Phi)\,, \quad V(\Phi) = g\big(\tfrac{1}{4}\mathrm{Tr}(\Phi^4) + \tfrac{1}{3}b\,\mathrm{Tr}(\Phi^3) + \tfrac{1}{2}c\,\mathrm{Tr}(\Phi^2)\big)\,,$$

where $g > 0$. Show that this theory has an $SO(3)$ symmetry. Let $\mathcal{M}_0 = \{\Phi_0 : V(\Phi_0) = V_{\min}\}$. Assume $SO(3)$ acts transitively on $\mathcal{M}_0$, i.e. all points in $\mathcal{M}_0$ can be linked by an $SO(3)$ transformation. Show that then all $\Phi_0 \in \mathcal{M}_0$ have the same eigenvalues, which add up to zero, and that we may choose Φ_0 so that it is diagonal. Describe how the eigenvalues of Φ_0 determine the unbroken subgroup of $SO(3)$.

For this theory show that $\mathcal{M}_0$ is determined by the equation

$$\Phi_0^3 + b\,\Phi_0^2 + c\,\Phi_0 = \mu\, I\,, \quad 3\mu = \mathrm{Tr}(\Phi_0^3) + b\,\mathrm{Tr}(\Phi_0^2)\,.$$

[Here μ may be regarded as a Lagrange multiplier for the condition $\mathrm{Tr}(\Phi) = 0$ when varying $V(\Phi)$.] Verify that there is a potential solution in which the unbroken subgroup is $SO(2)$ if $b^2 > 12c$. [Note that in this case Φ_0 may be given in terms of a single eigenvalue.]

For 3×3 traceless matrices $\mathrm{Tr}(M^4) = \frac{1}{2}\big(\mathrm{Tr}(M^2)\big)^2$. Show that if $b = 0$ the initial symmetry is in fact $SO(5)$ and that $V_{\min} = -\frac{1}{2}\,gc^2$ with an unbroken group $SO(4)$.

How do the results on possible unbroken symmetry groups generalise to the analogous theory with an $SO(N)$ symmetry defined in terms of $N\times N$ symmetric traceless matrices?

3. Consider an $SU(2)$ gauge theory coupled to a two component complex scalar field ϕ. The $SU(2)$ generators acting on ϕ are represented by $\boldsymbol{\tau} = \frac{1}{2}\boldsymbol{\sigma}$, with $\boldsymbol{\sigma}$ the usual Pauli matrices,

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\!\cdot\mathbf{F}_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \tfrac{1}{2}\lambda\big(\phi^\dagger\phi - \tfrac{1}{2}v^2\big)^2\,,$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - g\,\mathbf{A}_\mu\times\mathbf{A}_\nu\,, \quad D_\mu\phi = \partial_\mu\phi + \mathrm{i}g\,\mathbf{A}_\mu\cdot\boldsymbol{\tau}\phi\,,$$

$\mathbf{A}_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$ and $\mathbf{F}_{\mu\nu} = (F^1_{\mu\nu}, F^2_{\mu\nu}, F^3_{\mu\nu})$. [The cross product above arises because the $SU(2)$ structure constant is the Levi-Civita symbol: $[t^a, t^b] = i\epsilon^{abc}t^c$.] Explain why we may choose $\phi = (0, v{+}h)^T/\sqrt{2}$ and why the $SU(2)$ gauge symmetry is completely broken. Neglecting quantum corrections, what are the masses of the elementary particle states?

4. (i) A triplet gauge field $\mathbf{A}_\mu$ is coupled to a real triplet field $\boldsymbol{\phi}$ with the Lagrangian,

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\!\cdot\mathbf{F}_{\mu\nu} + \tfrac{1}{2}(D^\mu\boldsymbol{\phi})\cdot(D_\mu\boldsymbol{\phi}) - \tfrac{1}{8}\lambda\big(\boldsymbol{\phi}^2 - v^2\big)^2\,,\\ \mathbf{F}_{\mu\nu} &= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - e\,\mathbf{A}_\mu\times\mathbf{A}_\nu\,, \quad D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} - e\,\mathbf{A}_\mu\times\boldsymbol{\phi}\,.\end{aligned}$$

[I.e. $\boldsymbol{\phi}$ transforms in the adjoint representation of $SU(2)$. The cross product above arises from writing the $SU(2)$ generators in the adjoint representation as $(t^a)_{jk} = -i\epsilon_{ajk}$.] Show that this theory is invariant under $SU(2)$ gauge transformations but that this is broken by the ground state to $U(1)$. Rewrite the theory in terms of physical fields and determine their masses and couplings.

(ii) For a complex triplet field $\boldsymbol{\phi}$ suppose the Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}\,\mathbf{F}^{\mu\nu}\!\cdot\mathbf{F}_{\mu\nu} + (D^\mu\boldsymbol{\phi})\cdot(D_\mu\boldsymbol{\phi}) + \tfrac{1}{2}g^2(\boldsymbol{\phi}^*\times\boldsymbol{\phi})^2\,,$$

where $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \boldsymbol{\phi}^\dagger\boldsymbol{\phi}$. Show that in the classical ground state the potential may be minimised, up to a freedom of gauge transformations, by choosing $\boldsymbol{\phi}_0 = v\,\mathbf{e}_3/\sqrt{2}$ for any complex v where $\mathbf{e}_3$ is the unit vector in the 3-direction. Explain why $v \sim -v$ under residual gauge transformations. Why is it possible to impose the conditions $\mathrm{Re}(v^*\boldsymbol{\phi}\cdot\mathbf{e}_1) = \mathrm{Re}(v^*\boldsymbol{\phi}\cdot\mathbf{e}_2) = 0$? Determine the masses of the physical fields. Why are theories with different values of v^2 inequivalent?

5. A gauge theory for the group G is described by the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu}{}_a F_{\mu\nu a} + \tfrac{1}{2} (D^\mu\phi)\cdot(D_\mu\phi) - V(\phi)\,,$$
$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - g\, c_{abc} A_{\mu b} A_{\nu c}\,, \quad D_\mu\phi = \partial_\mu\phi + ig\, A_{\mu a}\theta_a\phi\,,$$

where ϕ is a multiplet of scalar fields, $a = 1,\ldots,\dim G$, and θ_a are matrices representing the Lie algebra of G, $[\theta_a, \theta_b] = ic_{abc}\theta_c$, where c_{abc} is completely antisymmetric. Assuming $V'(\phi)\cdot\theta_a\phi = 0$ and $\tilde{\phi}\cdot(\theta_a\phi) = (\widetilde{\theta_a\phi})\cdot\phi$, show that $\mathcal{L}$ is invariant under G gauge transformations. [Recall that $\tilde{\phi}\cdot\phi = \tilde{\phi}^\dagger\phi$.]

Suppose $V(\phi)$ is minimised at $\phi = \phi_0$ and that we add a gauge fixing term of the form

$$\mathcal{L}_{\text{g.f.}} = -\tfrac{1}{2}\big(\partial^\mu A_{\mu a} - g\,(i\theta_a\phi_0)\cdot\phi\big)\big(\partial^\nu A_{\nu a} - g\,\phi\cdot(i\theta_a\phi_0)\big)\,.$$

If $\phi = \phi_0 + f$ derive the decoupled linearised equations of motion for the vector and scalar fields,

$$\partial^2 A_{\mu a} + g^2(\theta_a\phi_0)\cdot(\theta_b\phi_0)\,A_{\mu b} = 0\,, \quad \partial^2 f + \mathcal{M}\cdot f + g^2(\theta_a\phi_0)\,(\theta_a\phi_0)\cdot f = 0\,,$$

where $\mathcal{M}$ is a matrix determined by the second derivatives of $V(\phi)$ at $\phi = \phi_0$. Show that the mass eigenstates form multiplets of the unbroken gauge group H (for which the corresponding gauge fields are massless). [It is sufficient to show that the mass matrices appearing in the linear field equations commute with the generators of H in the appropriate representation.]

6. Let $\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - \frac{1}{2}g(\phi^*\phi - \frac{1}{2}v^2)^2$ be the Lagrangian for a complex scalar field ϕ. Writing $\phi = (v + f + \mathrm{i}\alpha)/\sqrt{2}$, show that the α field is massless whereas the f field has a mass $\sqrt{gv^2}$. Consider the scattering amplitude $\mathcal{M}$ for α particle scattering which is defined by $\langle\alpha(p_3)\alpha(p_4)|T|\alpha(p_1)\alpha(p_2)\rangle = (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\mathcal{M}$ where the scattering S-matrix is $S = 1 - \mathrm{i}T$. Neglecting any Feynman diagrams with loops, show that

$$\mathcal{M} = g^2v^2\left(\frac{1}{s - gv^2} + \frac{1}{t - gv^2} + \frac{1}{u - gv^2}\right) + 3g\,,$$

where

$$s = (p_1 + p_2)^2\,,\ t = (p_3 - p_1)^2\,,\ u = (p_4 - p_1)^2\,.$$

Verify that $s + t + u = 0$ and hence show that for α particles with low energies E we have $\mathcal{M} = \mathrm{O}(E^4)$.

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